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Representations of Spacetime as Unitary Operation Classes; or Against the Monoculture of Particle Fields

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Spacetime is modeled as a homogeneous manifold given by the classes of unitary **U**(2) operations in the general complex operations $\mathbf{GL}(\mathbb{C}^2)$. The residual representations of this noncompact symmetric space of rank two are characterized by two continuous real invariants, one invariant interpreted as a particle mass for a positive unitary subgroup and the second one for an indefinite unitary subgroup related to nonparticle interpretable interaction ranges. Fields represent nonlinear spacetime $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ by their quantization and include necessarily nonparticle contributions in the timelike part of their flat-space Feynman propagator.

1. INTRODUCTION

1.1. Some Historical Remarks

Newton's interpretation of space and time as having an absolute ontology (two unaffected boxes wherein the physical objects play around) was by far more successful in the development of physical theories than Leibniz's opinion, who considered time and position space as relations, as labels to express their transformation properties. With Einstein the two boxes became one spacetime box affected by and affecting the gravitational interaction.

Weyl[11] made the first attempt to unify Einstein's gravity with Maxwell's electrodynamics by explaining the electromagnetic interactions as effected by fields which connect and compatibilize spacetime-dependent transformations from the noncompact Abelian dilatation group $\mathbf{D}(1) = \exp \mathbb{R}$. This gauge idea, used for the wrong patient, was made fruitful by switching over from

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the noncompact $\mathbf{D}(1)$ to the apparently right patient, the compact Abelian transformation group $\mathbf{U}(1) = \exp i\,\mathbb{R}$, a real Lie group defined in the complex. Therewith a dichotomy between external spacetime transformations with the Lorentz group $\mathbf{O}(1, 3)$ and internal unitary transformations comprising the electromagnetic group $\mathbf{U}(1)$ was established. The internal transformation group proliferated, the experimental and theoretical favorites being today the compact standard model interaction gauge groups $\mathbf{U}(1)$, $\mathbf{SU}(2)$, and $\mathbf{SU}(3)$ for hypercharge, isospin, and color, respectively.

General relativity and electrodynamics came originally in real formulations, whereas quantum theory with its "probability amplitudes," characteristic phase relations (transition elements), and $\mathbf{U}(1)$-invariant scalar product was born as a complex theory. The gauge approach ties the electromagnetic interaction to the $\mathbf{U}(1)$ phases of complex matter fields. The complex representation of the internal compact real Lie group does not fit easily in a real representation structure of the external transformations with the Poincaré group, i.e., the vector spaces $\mathbb{R}^3$ and $\mathbb{R}^4$ for position space and spacetime translations, respectively, acted on by the rotation and the Lorentz group, $\mathbf{O}(3)$ and $\mathbf{O}(1, 3)$, respectively. But complex representations came rather early also for the real spacetime transformations: The twofold split of a ray with silver atoms in the original Stern–Gerlach experiment was the starting point to replace the rotation group $\mathbf{SO}(3)$ with their real irreducible representation spaces, necessarily odd dimensional, e.g., real 3-dimensional position space, by its twofold covering spin group $\mathbf{SU}(2)$. For the Lorentz group, this entailed the transition to the complex represented real[2] Lie group $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$ covering the orthochronous group $\mathbf{SO}^+(1, 3)$. In a rather loose external–internal "unification" both groups, the Lorentz $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$ and the electromagnetic $\mathbf{U}(1)$ transformations, come together as subgroups of the full real 8-dimensional group $\mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})$ with a central correlation.(16) This group is represented directly by the transformations of charged spinor fields, e.g., of the right-handed lepton isosinglet fields in the standard model of electroweak and strong interactions. In contrast to such an external–internal unification of the Lorentz group with the Abelian hypercharge $\mathbf{U}(1)$ in $\mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})$ a unification of the non-Abelian groups $\mathbf{SU}(2)$ and $\mathbf{SU}(3)$ for isospin and color with $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$ remains an open problem.(17,18)

1.2. Equations of Motion?

The replacement of a finally oriented causality in Aristotelian physics by time derivative equations of motions with initial conditions was, as a

[2] The index $\mathbb{R}$ on the complex number $\mathbb{C}_{\mathbb{R}} = \mathbb{R} \oplus i\mathbb{R}$ indicates its use as a complex represented real structure. The complex 3-dimensional Lie group $\mathbf{SL}(\mathbb{C}^2$ and the real 6-dimensional one $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$ are kept apart by this—perhaps overcautious— notation.

method, the most important progress initiated by Newton. Subsequently, equations of motion were derived from Hamiltonians and Lagrangians using extremal principles. In the course of the last century Hamiltonians and Lagrangians were more and more motivated and constructed as invariants with respect to transformation Lie groups and Lie algebras.

In quantum mechanics the operational structures of physics come into full bloom[(5)]: The equations of motion can be interpreted as the transformations with the Lie group $\exp t \in \mathbf{D}(1)$, modeling time, expressed via the adjoint action $(d/dt)a = [iH, a]$ with a Hermitian Hamiltonian H giving a basis iH for the time translation Lie algebra[3] $\log \mathbf{D}(1) = \mathbb{R}$. The action of time and its diagonalization is an algebraic eigenvalue problem $[H, a] = E(a)a$—an equation of motion is its differential formulation only, i.e., $d/dt \cong i \text{ ad } H$.

In the characteristic example of a quantum harmonic oscillator the time action diagonalization gives integer energy eigenvalues. The involved representation of time $\mathbf{D}(1) \to \mathbf{U}(1)$ by a unitary group establishes the probability structure since $\mathbf{U}(1)$ is the invariance group of a scalar product for the complex representation space. Everything else, the definition of position and momentum as real linear combinations of creation and annihilation operators, which express the notion "linear duality," the construction of a Hilbert space with normalizable wave functions, etc., are formulations for the basic $\mathbf{U}(1)$-representation structure of time adapted for the description of experiments in the classical physics-oriented language.

The same procedure can be given explicitly, e.g., for the not so trivial nonrelativistic hydrogen atom as done by Fock[(4)] using the rotation-perihel invariance group of the Kepler dynamics, i.e., $\mathbf{SO}(4) \cong \mathbf{SU}(2) \times \mathbf{SU}(2)/\{\pm 1\}$ (compact, i.e., definite unitary) for bound states and $\mathbf{SO}^+(1, 3) \cong \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})/\{\pm 1\}$ (real, but indefinite unitary) for scattering states, to determine the Hamiltonian as invariant and to give the definite $\mathbf{U}(1)$ and indefinite $\mathbf{U}(1, 1)$ unitarity structure, respectively, of the time action representations.

So far in quantum field theory, a replacement of the equations of motion, e.g., in the standard model, by a purely algebraic transformation theory with eigenvalues and eigenvectors—as seems appropriate for a quantum theory—which can describe not only the scattering of particles, but also derive, in a bound-state structure, their existence and their properties in terms of eigenvalues, has not succeeded yet. In the following, I shall take some steps along this route.

[3] The Lie algebra for the Lie group G is denoted as $\log G$.

1.3. The Particle Prejudice

Relativistic quantum field theory is often praised as progress insofar as interactions and particles are unified—all interactions are parametrizable by particle fields. Even if such a viewpoint is qualified by extending the particle language also to off-shell energy-momenta, i.e, for mass-m particles to energy-momenta q with $q^2 \neq m^2$, it is simply not true. Apart from quarks and gluons as strong interaction parametrizing fields postulated without particle asymptotics (confinement), the most prominent example is the classical spin-less Coulomb interaction which comes in the quantum electromagnetic Lorentz vector field

$$\mathbf{A}(x) = \begin{pmatrix} \mathbf{A}_0 + \mathbf{A}_3 & \mathbf{A}_1 - i\mathbf{A}_2 \\ \mathbf{A}_1 + i\mathbf{A}_2 & \mathbf{A}_0 - \mathbf{A}_3 \end{pmatrix}(x)$$

with four components. The $\mathbf{SO}^+(1, 3)$-Lorentz vector properties with maximal Abelian subgroup $\mathbf{SO}(2) \times \mathbf{SO}^+(1, 1)$ leads to a unitary $\mathbf{U}(2) \times \mathbf{U}(1, 1)$ "metric." As seen in the harmonic analysis with energy-momentum-dependent creation and annihilation operators, only the two transversal components, related to a $\mathbf{U}(2)$-scalar product, are particle interpretable as left and right circularily polarized photons. From the remaining two components with indefinite $\mathbf{U}(1, 1)$-sesquilinear form one component is related to the gauge degree of freedom, and the last, fourth degree of freedom describes a quantum field interaction without particle parametrization.[15]

An unreflected one-to-one correspondence of quantum fields with particles is similar and somewhat related to a superficial naive interpretation of Lorentz transformations for spacetime translations

$$x = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

as blurring the difference between time and position space. Obviously, the situation is more subtle. Also in special relativity, timelike and spacelike translations $\det x = x^2 > 0$ and $x^2 < 0$, respectively, are Lorentz operation compatible concepts—they are clearly distinct, but no longer linear subspaces. The relativity of time translations $\mathbb{R}$ and position space translations $\mathbb{R}^3$ can be seen in the homogeneous nonlinear structure of the absolute concepts "timelike" and "spacelike." With the fixgroups ("little groups") $\mathbf{SO}(3)$, $\mathbf{SO}(1, 2)$, and the semidirect $\mathbf{SO}(2) \mathbin{\vec{\times}} \mathbb{R}^2$ for timelike, spacelike, and lightlike translations, respectively, and the dilatation group $\mathbf{D}(1) = \exp \mathbb{R}$ one has the nonlinear manifolds for the nontrivial spacetime translations

$$\begin{aligned} &\text{timelike future (past):} && \mathbf{D}(1) \times \mathbf{SO}^{+}(1,3)/\mathbf{SO}(3) \cong \mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{U}(2) \\ &\text{spacelike:} && \mathbf{D}(1) \times \mathbf{SO}^{+}(1,3)/\mathbf{SO}(1,2) \cong \mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{U}(1,1) \\ &\text{lightlike future (past):} && \mathbf{SO}^{+}(1,3)/\mathbf{SO}(2) \,\overline{\times}\, \mathbb{R}^2 \cong \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{U}(1) \,\overline{\times}\, \mathbb{C}_{\mathbb{R}} \end{aligned}$$

The properties of free particle fields are encoded in Feynman propagators,[4] e.g., for a Hermitian scalar particle field Φ with mass m

$$\langle \{\Phi, \Phi\}(x) - \epsilon(x_0)[\Phi, \Phi](x)\rangle = \frac{i}{\pi} \int \frac{d^4 g}{(2\pi)^3} \frac{1}{q^2 + io - m^2} e^{xiq}$$

with the on-shell quantization causally supported, i.e., $[\Phi, \Phi](x) = 0$ for $x^2 < 0$. If one uses a rest system in linear spacetime and, therewith, a basis for time translations in an obviously not Lorentz-compatible decomposition in time and position space, timelike translations $(x_0, \vec{x})$ with $x^2 > 0$ have in general also a nontrivial linear position space component $\vec{x}$. In relativistic field theories, the position space dependence of nonrelativistic interactions like the Yukawa or Coulomb interaction does not arise from spacelike translations, but from timelike ones $x^2 > 0$, in the example above from the off-shell causal contribution involving the principal value integration P:

$$\begin{aligned} \epsilon(x_0)[\Phi, \Phi](x) &= \int \frac{a^4 q}{(2\pi)^3} \epsilon(x_0 q_0) \delta(m^2 - q^2)\, e^{xiq} \\ &= \frac{1}{j\pi} \int \frac{d^4 q}{(2\pi)^3} \frac{1}{q_{\mathrm{P}}^2 - m^2} e^{xiq} \\ &\quad - 2i\pi \int dx_0\, \epsilon(x_0)\, [\Phi, \Phi]\, (x) = \frac{\exp(-|\vec{x}m|)}{|\vec{x}|} \end{aligned}$$

Only the on-shell Fock value of the quantization opposite commutator, in the example above

$$\langle \{\Phi, \Phi\}(x)\rangle = \int \frac{d^4 q}{(2\pi)^3} \delta(m^2 - q^2) e^{xiq}$$

which is also spacelike supported, is relevant for the asymptotic particle interpretation. The causally supported off-shell part $\epsilon(x_0)[\Phi, \Phi](x)$ in the Feynman propagator is a particle-related contribution to a more complicated spacetime representation structure, as will be elaborated in Section 3. I think that relativistic quantum theory using particle-related fields only is incomplete and unsatisfactory with respect to its causal spacetime representation content.

[4] The translation compatible shorthand (anti)commutator notation $[a, b]_{\pm}(x - y) = [a(y), b(x)]_{\pm}$ is used.

Table I

	Abelian	Non-Abelian	Eigenvalues, invariants
Compact	$\mathbf{U}(1)$	$\mathbf{U}(2)$	$\mathbb{Q}$
Noncompact	$\mathbf{GL}(\mathbb{C}_{\mathbb{R}})$	$\mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})$	$\mathbb{R}$
Homogeneous (noncompact)	$\mathbf{D}(1)$	$\mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{U}(2)$	$\mathbb{R}$

1.4. The Complication of Spacetime Theories

One may ask why an algebraization of spacetime theories is so difficult. One reason may be that in the double dichotomy "Abelian–non-Abelian" and "compact–noncompact" seen in parallel to the physical concepts, spacetime operations are both non-Abelian and noncompact (Tables I and II).

The nondecomposable representations of compact, Abelian transformations are complex one-dimensional, of compact-Nonabelian transformations complex finite-dimensional, both with rational eigenvalues[3,6]—as physical properties called, e.g., winding, charge, or spin numbers. As for the noncompact transformations, the irreducible representations in the Abelian case are still complex one-dimensional, in the non-Abelian case in general infinite-dimensional,[7,8] in both cases with a continuous spectrum—as physical properties called energies (frequencies), masses, or interaction ranges.

If we insist on the causality-compatible orthochronous Lorentz group $\mathbf{SO}^+(1,3)$ we have to face the representation complications of noncompact, non-Abelian transformations.

2. SPACETIME AS TRANSFORMATIONS

In this section a model for spacetime is proposed with spacetime points as classes of transformations.

2.1 A Mathematical Remark on "Naturalness"

In mathematics, there exist "natural" structures connected with the solution of "universal" problems[2] which may be superficially characterized as

Table II

	Abelian	Non-Abelian	Quantum numbers
Internal (compact)	Eletromagnetic	Electroweak	Winding, charge numbers, spin, multiplicities
External (noncompact)	Time	Spacetime	Frequencies, energies, masses, interaction ranges

follows: A given structure gives rise to new ones by considering its internal relations, e.g., its self-transformations as binary relations.

Some well-known elementary examples: Each Abelian semigroup with cancellation rule is naturally extendable to a unique group structure. This is used for the extension of the natural numbers to the integer ones as binary internal relations modulo an addition (+)-induced equivalence $\stackrel{+}{\sim}$,

$$\mathbb{Z} = \frac{\mathbb{N} \times \mathbb{N}}{\stackrel{+}{\sim}} \qquad \text{with} \quad (n_1, n_2) \stackrel{+}{\sim} (m_1, m_2) \Leftrightarrow n_1 + m_2 = m_1 + m_2$$

or for the extension of the integers as ring to the rationals as its unique field structure with a multiplication (·)-induced equivalence $\stackrel{\cdot}{\sim}$,

$$\mathbb{Q} = \frac{\mathbb{Z} \times [\mathbb{Z} \setminus \{0\}]}{\stackrel{\cdot}{\sim}} \qquad \text{with} \quad (z_1, z_2) \stackrel{\cdot}{\sim} (u_1, u_2) \Leftrightarrow z_1 u_2 = u_1 z_2$$

By considering Cauchy series as countably infinite relations, each metrical space has its unique, naturally Cauchy completed space. This is used for the extension of the rationals $\mathbb{Q}$ with their natural order-induced metric to the reals $\mathbb{R} = \mathbb{Q}^{\mathbb{N}_0} / \stackrel{C}{\sim}$ with a Cauchy series-induced equivalence $\stackrel{C}{\sim}$.

Another example is the natural structure of multilinearity: Each vector space gives rise to a unique unital associative algebra structure, its tensor algebra. Different quotient algebras can be related to the algebras used in classical and quantum theories.[14]

2.2. Adjoint Transformation Structures

Some natural concepts involving binary internal relations are called *adjoint*. They play a paramount role in physical theories, not only for gauge fields. With respect to real and complex Lie transformation groups and algebras (always finite-dimensional, if not stated explicitly otherwise) such adjoint concepts describe the action of the transformations on themselves and lead to characteristic doublings.

The adjoint doubling will be illustrated with the example of the real three-dimensional position space whose translations, formalized by a vector space $\mathbb{R}^3$, with the action of a rotation group $\mathbf{SO}(3)$ constitute a Euclidean semidirect product group

$$\mathbf{SO}(3) \overline{\times} \mathbb{R}^3 \qquad \text{with product} \quad (O_1, \vec{x}_1) \circ (O_2, \vec{x}_2) = (O_1 O_2, \vec{x}_1) + O_2(\vec{x}_2))$$

$\mathbf{SO}(3) \overline{\times} \mathbb{R}^3$ is an example of an *adjoint affine Lie group* where, in general, a Lie group G is represented in the automorphisms of the vector space structure of its Lie algebra log G,

$$G \times \log G \to \log G, \qquad (g, x) \mapsto \operatorname{Int} g(x) = g \circ x \circ g^{-1}$$
$$\operatorname{Int} g_1 \circ \operatorname{Int} g_2 = \operatorname{Int} g_1 g_2$$

The adjoint group representation is faithful for the *adjoint group* Int G, defined by the classes of the group elements with respect to the centrum, i.e., the kernel of the group representation Int,

$$\operatorname{Int} G \,\overline{\times}\, \underline{\log G}, \qquad \operatorname{Int} G = G / \operatorname{centr} G$$

$$\text{with product } (g_1, x_1) \circ (g_2, x_2) = (g_1 g_2, x_1 + \operatorname{Int} g_1(x_2))$$

The "linear underlining" of the Lie algebra $\underline{\log G}$ indicates that only its linear vector space structure is relevant for this adjoint doubling; the Lie bracket of the second factor has to be "forgotten."

The Euclidean group for position space, mentioned above, is the adjoint affine group of the spin group **SU**(2),

$$\mathbf{SO}(3) \,\overline{\times}\, \mathbb{R}^3 = \operatorname{Int} \mathbf{SU}(2) \,\overline{\times}\, \underline{\log \mathbf{SU}(2)}, \qquad \operatorname{centr} \mathbf{SU}(2) = \{\pm \mathbf{1}_2\}$$

$$\text{with product } (u_1, x_1) \circ (u_2, x_2) = (u_1 u_2, x_1 + u \circ x_2 \circ u^*)$$

Starting from the defining and fundamental complex two-dimensional Pauli **SU**(2)-representation by $u = \exp i\overline{\alpha}\overline{\sigma}$ with spin $J = 1/2$, which—up to equiavalence—gives all irreducible **SU**(2)-representations $[2J]$, $J = 0$, 1/2 1, . . . , with dimension $(1 + 2J)$ by totally symmetrical tensor products, one obtains the position space translations with the adjoint spin representation[(2)] as vector space structure of the spin Lie algebra, in a Leibnizian interpretation as binary relations (traceless linear mappings) for Pauli spinors:

$$\text{position space translations} = \underline{\log \mathbf{SU}(2)} \cong \mathbb{R}^3$$

$$\underline{\log \mathbf{SU}(2)} = \left\{x: \mathbb{C}^2_{\mathbb{R}} \to \mathbb{C}^2_{\mathbb{R}} \,\middle|\, \operatorname{tr} x = 0,\ x = x^* = \overline{\sigma}\overline{x}\right.$$
$$\left. = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}\right\}$$

The rotations are realized by the adjoint representation

$$u \in \mathbf{SU}(2): \quad x \mapsto u \circ x \circ u^* = O(u)(x)$$

The position space metric comes as a negative-definite Killing form, inherited from the spin Lie algebra, i.e., the **SU**(2)-invariant double trace $\operatorname{tr} x \circ y$, with the quadratic form as the determinant $x^2 = 1/2 \operatorname{tr} x \circ x = -\det x$.

For a Lie algebra L, the *adjoint affine Lie algebra* is defined by the adjoint representation which realizes the Lie bracket by the commutator of the endomorphisms of its vector space structure:

$$L \times L \to L, \qquad (l, x) \mapsto \text{ad}\, l(x) = [l, x]$$

$$\text{ad}[l_1, l_2] = [\text{ad}\, l_1, \text{ad}\, l_2]$$

Only the adjoint Lie algebra ad L given by the classes of the Lie algebra with respect to the centrum is faithfully represented. The adjoint affine Lie algebra is as vector space the direct sum ad $L \oplus L$ and as Lie algebra the semidirect bracket product, denoted by the direct sum–semidirect Lie bracket symbol $\overline{\oplus}$,

$$\text{ad}\, \underline{L}\, \overline{\oplus}\, \underline{L} = \{l + x \,|\, l, x \in \underline{L}\}, \qquad \text{ad}\, \underline{L} = \underline{L}/\text{centr}\, \underline{L}$$

$$\text{with bracket } [l_1 + x_1, l_2 + x_2] = [l_1, l_2] + \text{ad}\, l_1(x_2) - \text{ad}\, l_2(x_1)$$

The second factor $\underline{L}$ in this adjoint doubling is the vector space structure of the Lie algebra.

For Euclidean position space, the adjoint affine Lie algebra for the angular momenta log $\mathbf{SO}(3)$ is the Lie algebra of the Euclidean group

$$\log\mathbf{SO}(3)\, \overline{\oplus}\, \mathbb{R}^3 \cong \log\mathbf{SU}(2)\, \overline{\oplus}\, \underline{\log \mathbf{SU}(2)}$$

Both adjoint doublings, the adjoint affine group and the adjoint affine Lie algebra, are related to the realization of a group G on itself by inner automorphisms

$$G \times G \to G, \qquad (g, a) \mapsto \text{Int}\, g(a) = gag^{-1}$$

$$\text{Int}\, g_1 \circ \text{Int}\, g_2 = \text{Int}\, g_1 g_2$$

leading to the *adjoint group doubling* as the semidirect product

$$\text{Int}\, G\, \overline{\times}\, G = \{(g, a) \,|\, g, a \in G\}$$

$$\text{with product } (g_1, a_1) \circ (g_2, a_2) = (g_1 g_2, a_1\, \text{Int}\, g(a_2))$$

Each semidirect group $\underline{G'}\, \overline{\times}\, G$ is isomorphic to a subgroup of the adjoint group doubling Int $G \times G$, which is universal in this sense.

The adjoint doubling of the spin group is its semidirect product with the rotation group

$$\mathbf{SO}(3)\, \overline{\times}\, \mathbf{SU}(2)$$

2.3. Spacetime and the Causal Poincaré Group

For a Lie group G with Lie algebra log G one has the three semidirect adjoint doublings, reflecting two steps of infinitesimalization (Table III).

Table III

	Name	Example
Int $G \overline{\times} G$	Adjoint group doubling	$\mathbf{SO}(3) \overline{\times} \mathbf{SU}(2)$
Int $G \times \underline{\log G}$	Adjoint affine group	$\mathbf{SO}(3) \times \mathbb{R}^3$
log Int $G \overline{\oplus} \underline{\log G}$	Adjoint affine Lie algebra	$\log \mathbf{SO}(3) \overline{\oplus} \mathbb{R}^3$

They were discussed in the last section for the unitary spin group $u^* = u^{-1} \in \mathbf{SU}(2)$ with the rotations $\mathbf{SO}(3)$ as adjoint group and the position space translations $\mathbb{R}^3$ as vector space structure of the spin Lie algebra log $\mathbf{SU}(2)$.

What about spacetime? The spacetime translations $\mathbb{R}^4$ (Minkowski space) with the orthochronous Lorentz group action constitute the semidirect product Poincaré group

$$\mathbf{SO}^+(1, 3) \overline{\times} \mathbb{R}^4$$

The Poincaré group is not the adjoint affine Lie group of the real six-dimensional Lie group $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$:

$$\text{Int } \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}}) \overline{\times} \underline{\log \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})} = \mathbf{SO}^+(1, 3) \overline{\times} \mathbb{R}^6$$

$$\text{Int } \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}}) = \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})/\{\pm \mathbf{1}_2\} \cong \mathbf{SO}^+(1, 3)$$

This real 12-dimensional group is relevant for the gauge structures in Minkowski space where the curvature fields, e.g., the electromagnetic field strengths $\{F^{jk} = -F^{kj}\}_{j,k=0}^3 = \{\vec{E}, \vec{B}\}$, represent the real six-dimensional vector space structure of the Lorentz Lie algebra with the adjoint Lorentz group action.

At first sight it seems unnatural to relate the real four-dimensional Minkowski translations $\mathbb{R}^4$ to the real six-dimensional Lorentz Lie algebra $\log \mathbf{SO}^+(1, 3) \cong \log \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$. However, it is exactly the complex representation of the real covering group $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$ which makes this relation natural in the mathematical sense. Only in this context can the Poincaré group for flat spacetime be understood as arising from an adjoint doubling, i.e., related to internal relations of a transformation group.

In the case of complex represented real transformations there are two kinds of adjoint structures. It may be helpful to give the construction first in abstract terms: If a semigroup G has a reflection (conjugation), i.e., an involutive contra-automorphism defined by

$$*: \quad G \to G, \qquad g^{**} = g, \qquad (gh)^* = h^* g^*$$

it defines its $*$-*symmetric domain* as the subset

$$D(G) = \{d \in G \big| d^* = d\}$$

The concatenation of the inversion of a group G as canonical group reflection with any reflection (conjugation) $*$ is an involutive automorphism

$$\hat{}: \quad G \to G, \qquad \hat{g} = (g^{-1})^* = (g^*)^{-1}$$

The invariants for this automorphism constitute the $*$-unitary subgroup

$$U(G) = \{u \in G \big| u^{-1} = u^*\}$$

For a group with conjugation both the symmetric domain $D(G)$ and the unitary subgroup $U(G)$ can be used for adjoint structures.

Physically relevant examples used in the following are the full general complex linear groups $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})$, considered as real Lie groups and definable by the nonsingular complex $n \times n$ matrices with the Hermitian matrix conjugation $*$. They will be used for time in the case $n = 1$ and for spacetime with $n = 2$. The group $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})$ has the real n^2-dimensional submanifold $\mathbf{D}(n)$ as its symmetric domain and the real n^2-dimensional Lie subgroup $\mathbf{U}(n)$ as the group with the invariants

$$\mathbf{D}(n) = \{d \in \mathbf{GL}(\mathbb{C}^n_{\mathbb{R}}) \big| d^* = d\}, \qquad \mathbf{U}(n) = \{u \in \mathbf{GL}(\mathbb{C}^n_{\mathbb{R}}) \big| u^* = u^{-1}\}$$

The symmetric domain is a symmetric space[9] with the maximal compact group as fixgroup. It is the direct product of the Abelian group $\mathbf{D}(\mathbf{1}_n) = \mathbf{1}_n \exp \mathbb{R}$ and the globally symmetric space $\mathbf{SD}(n) = \mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbf{SU}(n)$,

$$\mathbf{D}(n) \cong \mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})/\mathbf{U}(n) \cong \mathbf{D}(\mathbf{1}_n) \times \mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbf{SU}(n)$$

Back to the general structure: A group G with two reflections, $g \leftrightarrow g^*$ (conjugation) and $g \leftrightarrow g^{-1}$ (inversion), gives rise to two types of adjoint doublings. The inversion-induced inner automorphisms of the group G described in the former section

$$G \times G \to G, \qquad (g, a) \mapsto \text{Int } g(a) = gag^{-1} = (\hat{g}a^*\hat{g}^*)^*$$

$$\text{Int } g_1 \circ \text{Int } g_2 = \text{Int } g_1 g_2, \qquad \text{kern Int} G = \text{centr } G$$

are, in general, not compatible with the conjugation. In addition and in analogy to the inner automorphisms Int, the group G allows the conjugation-compatible bijections, denoted by Int_*:

$$G \times G \to G, \qquad (g, a) \mapsto \text{Int}_* \, g(a) = gag^* = (ga^*g^*)^*$$

$$\text{Int}_* \, g_1 \circ \text{Int}_* \, g_2 = \text{Int}_* \, g_1 g_2$$

Also these bijections constitute a realization of the group G with the kernel defining the faithfully realized classes $\text{Int}_* \, G$:

$$\text{kern Int}_* = \{h \in G \big| hgh^* = h \text{ for all } g \in G\}$$
$$\text{Int}_* \, G = G/\text{kern Int}_*$$

For the unitary elements $u \in U(G)$ the bijections coincide with the inner automorphisms, i.e., $\text{Int}_* \, u = \text{Int } u$, not, however, in general. The analogous structure to the adjoint group doubling $\text{Int } G \times G$ is given by the action of the conjugation-compatible bijections on the symmetric domain $D(G)$, which will be called the *adjoints symmetric transformaton space*:

$$\text{Int}_* \, G \,\overline{\times}\, {}_*D(G)$$

which, in general in contrast to $\text{Int } G \,\overline{\times}\, G$, is no semidirect product group.

The two types of adjoint doublings are illustrated for the physically relevant examples $\mathbf{GL}(\mathbb{C}^n)$: One obtains for the complex case $\mathbb{C}$ with the inversion and for the complex represented real one $\mathbb{C}_\mathbb{R}$ with the conjugation

$$\text{inversion:} \quad \text{Int } \mathbf{GL}(\mathbb{C}^n) = \mathbf{GL}(\mathbb{C}^n)/\mathbf{GL}(\mathbb{C}) = \mathbf{SL}(\mathbb{C}^n)/\mathbb{I}(n)$$
$$\text{conjugation:} \quad \text{Int}_* \, \mathbf{GL}(\mathbb{C}^n_\mathbb{R}) = \mathbf{GL}(\mathbb{C}^n_\mathbb{R})/\mathbf{U}(\mathbf{1}_n) = \mathbf{D}(\mathbf{1}_n) \times \mathbf{SL}(\mathbb{C}^n_\mathbb{R})/\mathbb{I}(n)$$

with the cyclotomic group $\mathbb{I}(n) = \{z \in \mathbb{C} \big| z^n = 1\}$ as $\mathbf{SL}(\mathbb{C}^n)$-centrum. This leads to the adjoint group doublings and the adjoint symmetric transformation spaces

$$n = 1: \begin{cases} \text{inversion:} & \text{Int } \mathbf{GL}(\mathbb{C}) \,\overline{\times}\, \underline{\mathbf{G}}\mathbf{L}(\mathbb{C}) = \mathbf{GL}(\mathbb{C})\_\_ \\ \text{conjugation:} & \text{Int}_* \, \mathbf{GL}(\mathbb{C}_\mathbb{R}) \,\overline{\times}_*\, \mathbf{D}(1) = \mathbf{D}(1) \,\overline{\times}\, {}_*\mathbf{D}(1) \end{cases}$$

$$n = 2: \begin{cases} \text{inversion:} & \text{Int } \mathbf{GL}(\mathbb{C}^2) \,\overline{\times}\, \underline{\mathbf{G}}\mathbf{L}(\mathbb{C}^2) = \mathbf{SL}(\mathbb{C}^2)/\mathbb{I}(2) \,\overline{\times}\, \mathbf{GL}(\underline{\mathbb{C}^2}) \\ \text{conjugation:} & \text{Int}_* \, \mathbf{GL}(\mathbb{C}^2_\mathbb{R}) \,\overline{\times}\, {}_*\mathbf{D}(2) = [\mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}^+(1, 3)] \,\overline{\times}\, {}_*\mathbf{D}(2) \end{cases}$$

For spacetime with $n = 2$ the conjugate adjoint action involves the direct product of the orthochronous Lorentz group and the dilatation group $\mathbf{D}(\mathbf{1}_2)$, called the causal group in this context.

Obviously for Lie symmetries the adjoint Lie group structures are linearizable with Lie algebra structures, first in general: For a Lie group G with reflection (conjugation), the Lie algebra $\log G$ inherits the reflection (conjugation). Therefore, it is the direct sum of the $*$-antisymmetrical Lie subalgebra $l^* = -l$ as Lie algebra of the unitary Lie subgroup and the isomorphic $*$-symmetrical vector subspace $x = +x^*$ as tangent structure of the symmetric manifold $D(G) = G/U(G)$,

$$\log G = \log G_- \oplus \log G_+, \qquad \begin{cases} \log G_- = \log U(G) \\ \log G_+ \cong \log G/\log U(G) \end{cases}$$

In the example above one has in addition to the Lie algebra $\log \mathbf{U}(n)$ as $\mathbf{U}(n)$-tangent space a real n^2-dimensional vector subspace $\mathbb{R}(n)$ as tangent space of the symmetrical domain $\mathbf{D}(n)$ which, for $n = 2$, will be used as spacetime translations (Minkowski space)

$$\log \mathbf{GL}(\mathbb{C}^n_{\mathbb{R}}) = \log \mathbf{U}(n) \oplus \mathbb{R}(n)$$

$$\mathbb{R}(n) \cong \log \mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})/\log \mathbf{U}(n) \cong \mathbb{R}^{n^2}$$

In addition to the adjoint affine Lie group Int $G \,\overline{\times}\, \underline{\log G}$ involving the inversion as natural reflection one has now also the conjugate adjoint representation of the group on its Lie algebra,

$$G \times \log G \to \log G, \qquad (g, m) \mapsto \mathrm{Int}_* g(m) = g \circ m \circ g^* = (g \circ m^* \circ g^*)^*$$

$$\mathrm{Int}_* g_1 \circ \mathrm{Int}_* g_2 = \mathrm{Int}_*\, g_1 g_2$$

which, with the conjugation compatibility, can be restricted to the symmetrical and antisymmetrical vector subspaces of log G. The *conjugate adjoint affine Lie group* is defined with the symmetrical subspace as translations,

$$\mathrm{Int}_* G \,\overline{\times}_*\, \log G_+$$

With respect to the two adjoint doublings Int $G \,\overline{\times}\, \underline{\log G}$ with inversion and $\mathrm{Int}_* G \,\overline{\times}_*\, \log G_+$ with conjugation one obtains in the spacetime relevant example ($n = 2$) for the second case the Poincaré group with an additional causal group action:

$$n = 1: \begin{cases} \text{inversion:} & \mathrm{Int}\, \mathbf{GL}(\mathbb{C}) \,\overline{\times}\, \underline{\log \mathbf{GL}(\mathbb{C})} = \underline{\mathbb{C}} \\ \text{conjugation:} & \mathrm{Int}_* \mathbf{GL}(\mathbb{C}_{\mathbb{R}}) \,\overline{\times}_*\, \mathbb{R}(1) = \mathbf{D}(1) \,\overline{\times}_*\, \mathbb{R} \end{cases}$$

$$n = 2: \begin{cases} \text{inversion:} & \mathrm{Int}\, \mathbf{GL}(\mathbb{C}^2) \,\overline{\times}\, \underline{\log \mathbf{GL}(\mathbb{C}^2)} = \mathbf{SL}(\mathbb{C}^2)/\mathbb{I}(2) \,\overline{\times}\, \underline{\mathbb{C}^4} \\ \text{conjugation:} & \mathrm{Int}_* \mathbf{GL}(\mathbb{C}^2_{\mathbb{R}}) \,\overline{\times}_*\, \mathbb{R}(2) = [\mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}^+(1, 3)] \,\overline{\times}_*\, \mathbb{R}^4 \end{cases}$$

All finite-dimensional irreducible complex $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$-representations $[2L|2R]$ with half-integers $L, R = 0, 1/2, 1, \ldots$ and dimension $(1 + 2L)(1 + 2R)$ can be built—up to equivalence—by the totally symmetrical tensor products of the two fundamental Weyl representations, related to each other by the conjugation-induced automorphism

$$\begin{array}{lll} \text{left-handed} & [1|0] \text{ by} & s = \exp(+\vec{\beta} + i\vec{\alpha})\vec{\sigma} \\ \text{right-handed} & [0|1] \text{ by} & \hat{s} = \exp(-\vec{\beta} + i\vec{\alpha})\vec{\sigma} \end{array}$$

The representations have the conjugation $[2L|2R]^* = [2R|2L]$. The Hermitian irreducible $(1 + 2J)^2$-dimensional representations $[2J|2J]$ with $J = 0, 1/2, 1, \ldots$ are generated by the complex four-dimensional Minkowski $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$-representation $[1|1] = [1|0] \otimes [0|1]$ with the linear binary relations for Weyl spinors. The symmetric (real) subspace is the Cartan representation of the spacetime translations by linear spinor mappings with the Weyl matrices $\sigma^k \cong (\mathbf{1}_2, \vec{\sigma})$

$$\text{spacetime translations} = \log \mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})_+ = \mathbb{R}(2) \cong \log \mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})/\log \mathbf{U}(2)$$

$$\mathbb{R}(2) = \left\{x: \mathbb{C}^2_{\mathbb{R}} \to \mathbb{C}^2_{\mathbb{R}} \middle| x = x^* = \sigma^k x_\kappa = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + x_2 & x_0 - x_3 \end{pmatrix}\right\} \cong \mathbb{R}^4$$

They are acted on with the Lorentz and the causal group by the conjugate adjoint representation

$$s \in \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}}): \quad x \mapsto s \circ x \circ s^* = \Lambda(s)(x)$$

$$d = d^* \in \mathbf{D}(\mathbf{1}_2): \quad x \mapsto d \circ x \circ d^* = D(d)(x)$$

The Lorentz metric comes as a product $g = \epsilon \otimes \epsilon^{-1}$ with the invariant spinor metric, i.e., the antisymmetric bilinear $\mathbb{C}^2$-volume form $\epsilon = -\epsilon^T$, leading to the indefinite signature sign $g = (1, 3)$.

2.4. Spacetime as Unitary Operation Classes

We summarize the salient structures of the last section which will be used in the following. The conjugate adjoint operation structure for the group $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})$ suggests the definition of nonlinear models for time and spacetime as symmetric domains for complex linear transformations where the spacetime points are the complex linear operations modulo the maximal compact unitary operation group;

$$\mathbf{D}(n) = \mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})/\mathbf{U}(n)$$

$$\text{time } (n = 1): \quad \mathbf{D}(1) = \exp \mathbb{R}$$

$$\text{spacetime } (n = 2): \quad \mathbf{D}(2) \cong \mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$$

Time comes as group, spacetime as homogeneous manifold. $\mathbf{D}(n)$ is the orientation manifold[11] of scalar products in n dimension.[16]

The translations are the corresponding tangent structures:

$$\mathbb{R}(n) = \log \mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})/\log \mathbf{U}(n) \cong \mathbb{R}^{n^2}$$

$$\text{time translations } (n = 1): \quad \mathbb{R}(1) = \mathbb{R}$$

$$\text{spacetime translations } (n = 2): \quad \mathbb{R}(2) \cong \mathbb{R} \oplus \log \mathbf{SO}^+(1, 3)/\log \mathbf{SO}(3)$$

As subsets of the complex $(n \times n)$-matrices which constitute a stellar algebra, time and spacetime carry the spectrum-induced order, i.e., the natural order for time and the Minkowski partial order for spacetime:

$$x = x^* = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \text{positive}$$

$$\Leftrightarrow \text{spec } x \geq 0$$

$$\Leftrightarrow x^2 = \det x \geq 0, \qquad x_0 = 1/2 \operatorname{tr} x \geq 0$$

$$\Leftrightarrow x = \vartheta(x^2)\epsilon(x_0)x$$

The conjugate adjoint affine group is the semidirect causal Poincaré group

$$[\mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}^+(1, 3)] \,\overline{\times}_* \,\mathbb{R}(2)$$

Here in the conjugate adjoint doubling, the causal structure and the boost structure arise twice—globally as $\mathbf{D}(\mathbf{1}_2)$ and $\mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$ and in the tangent space $\mathbb{R}(2) \cong \mathbb{R} \oplus \mathbb{R}^3$ as time and position space translations where the decomposition is incompatible with the $\mathbf{SO}^+(1, 3)$-action.

3. REPRESENTATIONS OF SPACETIME

In analogy to Lie groups and algebras, also spacetime in the symmetric space model $\mathbf{D}(2) = \mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{U}(2)$ has linear representations. These representations will be constructed as residues in analogy to the representations of time, modeled by the group $\mathbf{D}(1) = \exp \mathbb{R}$, which is used in the quantization of the basic quantum mechanical dual pair "position-momentum."

3.1. Quantum Representations of Time

A dynamics is a representation of time, expressed in quantum mechanics by the noncommutativity of the generating operators. In the simplest cases of a harmonic oscillator or of a free mass point one has the time-dependent commutation relations of the dual position-momentum pair $(\mathbf{x}, \mathbf{p})$ which generates the operator algebra

$$\begin{pmatrix} [i\mathbf{p},\mathbf{x}] & [\mathbf{x},\mathbf{x}] \\ [\mathbf{p},\mathbf{p}] & [\mathbf{x},-i\mathbf{p}] \end{pmatrix}(t) = \begin{cases} D\left(\dfrac{t}{M}\middle| m^2\right) = \begin{pmatrix} \cos tm & i/Mm \sin tm \\ iMm \sin tm & \cos tm \end{pmatrix} \\ D\left(\dfrac{t}{M}\middle| 0\right) = \begin{pmatrix} 1 & it/M \\ 0 & 1 \end{pmatrix} \end{cases}$$

oscillator mass M and frequency m free point mass M) with the shorthand notation $[a(s), b(t)] = [a, b](t - s)$, valid for all matrix elements.

The time translations which generate the $\mathbf{D}(1)$-representation are quantum represented with the Hamiltonian, e.g., for the harmonic oscillator with creation and annihilation operators (u, u*)

$$H = \frac{\mathbf{p}^2}{2M} + \frac{m^2 M}{2}\mathbf{x}^2 = m\frac{\{\mathrm{u}, \mathrm{u}^*\}}{2}, \qquad \mathbf{u} = \frac{Mm\mathbf{x} - i\mathbf{p}}{\sqrt{2Mm}}$$

$$\mathbf{D}(1) \ni e^t \mapsto [\mathrm{u}^*, \mathrm{u}](t) = e^{tim} \in \mathbf{U}(1)$$

The harmonic oscillator $\mathbf{D}(1)$-representation by positions–momentum is decomposable into two irreducible representations in $\mathbf{U}(1) \ni e^{\pm tim}$, dual to each other with the $\mathbf{SO}(2)$-metric $\begin{pmatrix} 1/Mm & 0 \\ 0 & Mm \end{pmatrix}$ built with the intrinsic oscillator length $l^2 = 1/Mm$,

$$\mathbf{D}(1) \ni e^t \mapsto \begin{pmatrix} \cos tm & i/Mm \sin tm \\ iMm \sin tm & \cos tm \end{pmatrix} \cong \begin{pmatrix} e^{+tim} & 0 \\ 0 & e^{-tim} \end{pmatrix} \in \mathbf{SO}(2)$$

In contrast to the positive unitary time representations, not faithful for the simply connected group $\mathbf{D}(1)$, the free mass point is a faithful and reducible, but nondecomposable complex $\mathbf{D}(1)$-representation[1,13] in a noncompact indefinite unitary group,

$$\mathbf{D}(1) \ni e^t \mapsto \begin{pmatrix} 1 & it/M \\ 0 & 1 \end{pmatrix} \in \mathbf{U}(1, 1)$$

For the general quantum mechanical case with the Hamiltonian $iH = i[\mathbf{p}^2/2M + V(\mathbf{x})]$ as basis for the represented Lie algebra $\log \mathbf{D}(1) \cong \mathbb{R}$ one obtains the time $\mathbf{D}(1)$-representation by the ground-state values $\langle [a(s), b(t)] \rangle = \langle [a, b] \rangle (t - s)$ of the commutators with a spectral measure $\mu(m^2)$ for the time translation eigenvalues $m \in \mathbb{R}$ (frequencies, energies). In the case of a compact time development, where there exists a basis of normalizable energy eigenvectors (for the oscillator built by the monomials of creation and annihilation operators, the $\mathbf{D}(1)$-representation reads, with a positive-definite energy measure $\mu(m^2) \geq 0$,

$$\left\langle \begin{pmatrix} [i\mathbf{p}, \mathbf{x}] & [\mathbf{x}, \mathbf{x}] \\ [\mathbf{p}, \mathbf{p}] & [\mathbf{x}, -i\mathbf{p}] \end{pmatrix} \right\rangle (t) = \int_0^\infty dm^2 \, \mu(m^2) \begin{pmatrix} \cos tm & (i/Mm) \sin tm \\ iMm \sin tm & \cos tm \end{pmatrix}$$

3.2. The Representation Defect of Particle Fields

Particle fields are appropriate to describe *free* particles, i.e., representations of the spacetime tangent structures leading to the particle characterization by a causal translation property mass $m \neq 0$ or $m = 0$ with a rotation property $\mathbf{SU}(2)$-spin and a $\mathbf{U}(1)$-polarization, respectively.[12] What about representations of the nonlinear global spacetime model

$$\mathbf{D}(2) \cong \mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$$

which contains the rotation classes of the Lorentz transformations in addition to the causal group?

An appropriate example is a Dirac field $\boldsymbol{\Psi}$ for a particle with nontrivial mass m, e.g., for the electron–positron, with the quantization

$$\{\overline{\Psi}, \Psi\}(x) = \int \frac{d^4q}{(2\pi)^3} \epsilon(q_0)(\gamma^k q_k \pm m)\delta(q^2 - m^2)e^{xiq}$$
$$= \gamma^0 \delta(\vec{x}) \qquad \text{for} \quad x_0 = 0$$

causally supported, i.e., $\{\overline{\Psi}, \Psi\}(x) = 0$ for $x^2 < 0$.

The Dirac field is decomposable into left-and right-handed parts with the Weyl matrices $\sigma^k \cong (\mathbf{1}_2, \vec{\sigma}) \cong \check{\sigma}_k$,

$$\Psi(x) = \mathbf{l}(x) \oplus \mathbf{r}(x), \qquad \overline{\Psi}(x) = \Psi^*(x)\gamma^0 = \mathbf{r}^*(x) \oplus \mathbf{l}^*(x)$$

The field quantization

$$\gamma_0\{\overline{\Psi}, \Psi\}(x) = \begin{pmatrix} \{\mathbf{l}^*, \mathbf{l}\} & \{\mathbf{r}^*, \mathbf{l}\} \\ \{\mathbf{l}^*, \mathbf{r}\} & \{\mathbf{r}^*, \mathbf{r}\} \end{pmatrix}(x)$$
$$= \int \frac{d^4q}{(2\pi)^3} \epsilon(q_0) \begin{pmatrix} q_k\check{\sigma}_0\sigma^k & m\check{\sigma}_0 \\ m\sigma_0 & q_k\sigma_0\check{\sigma}^k \end{pmatrix} \delta(q^2 - m^2)e^{xiq}$$
$$\partial_k\check{\sigma}^k\mathbf{l}(x) = im\mathbf{r}(x), \qquad \partial_k\sigma^k\mathbf{r}(x) = im\mathbf{l}(x)$$

has to be compared with the energy spectral representation of the quantum mechanical time representation for the harmonic oscillator,

$$\begin{pmatrix} [i\mathbf{p}, \mathbf{x}] & [\mathbf{x}, \mathbf{x}] \\ [\mathbf{p}, \mathbf{p}] & [\mathbf{x}, -i\mathbf{p}] \end{pmatrix}(t) = \int dE\, \epsilon(E) \begin{pmatrix} E & 1/M \\ Mm^2 & E \end{pmatrix} \delta(E^2 - m^2)e^{tiE}$$
$$= \begin{pmatrix} \cos tm & (i/Mm) \sin tm \\ iMm \sin tm & \cos tm \end{pmatrix} = \mathbf{1}_2 \qquad \text{for} \quad t = 0$$
$$\frac{d}{dt}\mathbf{x}(t) = \frac{1}{M}\mathbf{P}(t), \qquad \frac{d}{dt}\mathbf{P}(t) = -Mm^2\mathbf{x}(t)$$

Particle fields give a causally supported position-space distribution of a time group $\mathbf{D}(1)$-representation as seen in the position space integral (time projection) of the quantization condition for a Dirac particle field

$$\int d^3x\, \gamma_0\{\overline{\Psi}, \Psi\}(x) = \int d^3x \begin{pmatrix} \{\mathbf{l}^*, \mathbf{l}\} & \{\mathbf{r}^*, \mathbf{l}\} \\ \{\mathbf{l}^*, \mathbf{r}\} & \{\mathbf{r}^*, \mathbf{r}\} \end{pmatrix}(x)$$
$$= \int dE\, \epsilon(E) \begin{pmatrix} E\mathbf{1}_2 & m\mathbf{1}_2 \\ m\mathbf{1}_2 & E\mathbf{1}_2 \end{pmatrix} \delta(E^2 - m^2)\, e^{x_0 iE}$$
$$= \begin{pmatrix} \cos x_0 m\, \mathbf{1}_2 & i \sin x_0 m\, \mathbf{1}_2 \\ i \sin x_0 m\, \mathbf{1}_2 & \cos x_0 m\, \mathbf{1}_2 \end{pmatrix}$$

where the momentum and position space integrations have been interchanged. For every time x_0 the position space integration goes over a compact sphere $\{\vec{x}|\vec{x}^2 \leq x_0^2\}$.

The integration with respect to the time translations displays the Yukawa interaction and force,

$$
\begin{aligned}
&2\pi \int dx_0\, \epsilon(x_0)\gamma_0\{\overline{\Psi}, \Psi\}(x) \\
&= 2\pi \int dx_0\, \epsilon(x_0)\begin{pmatrix} \{\mathbf{l}^*, \mathbf{l}\} & \{\mathbf{r}^*, \mathbf{l}\} \\ \{\mathbf{l}^*, \mathbf{r}\} & \{\mathbf{r}^*, \mathbf{r}\} \end{pmatrix}(x) \\
&= \int dQ \begin{pmatrix} |Q|\vec{\sigma}\vec{x}/|\vec{x}| & im\mathbf{1}_2 \\ im\mathbf{1}_2 & -|Q|\vec{\sigma}\vec{x}/|\vec{x}| \end{pmatrix} \vartheta(Q^2 - m^2)\, e^{-|\vec{x}Q|} \\
&= \begin{pmatrix} \frac{1 + |\vec{x}m|}{|\vec{x}|}\frac{\vec{\sigma}\vec{x}}{|\vec{x}|} & im\,\mathbf{1}_2 \\ im\,\mathbf{1}_2 & -\frac{1 + |\vec{x}m|}{|\vec{x}|}\frac{\vec{\sigma}\vec{x}}{|\vec{x}|} \end{pmatrix} \frac{e^{-|\vec{x}m|}}{|\vec{x}|}
\end{aligned}
$$

The rank 1 homogeneous boost manifold $\mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$ contains as maximal Abelian subgroup the Lorentz transformations $\mathbf{SO}^+(1, 1)$ isomorphic to a dilatation group $\mathbf{D}(1)$ with representations characterized by a mass (inverse length) m

$$
\mathbf{D}(1) \ni e^x \mapsto \begin{pmatrix} \cosh xm & \sinh xm \\ \sinh xm & \cosh xm \end{pmatrix} \cong \begin{pmatrix} e^{+xm} & 0 \\ 0 & e^{-xm} \end{pmatrix} \in \mathbf{SO}^+(1, 1)
$$

Particle fields involve representations only for the time group $e^{x_0} \in \mathbf{D}(1)$, but not for the Abelian boost group $e^{\pm|\vec{x}|} \in \mathbf{SO}^+(1, 1)$ as seen in the quantization of the left-handed Weyl field $\mathbf{l}(x)$:

$$
1/2 \text{ tr} \int d^3x\, \{\mathbf{l}^*, \mathbf{l}\}(x) = \int dE\, \epsilon(E)E\delta(E^2 - m^2)\, e^{x_0 iE} = \cos x_0 m
$$

$$
\begin{aligned}
\pi \text{ tr} \frac{\vec{\sigma}\vec{x}}{|\vec{x}|}\int dx_0\, \epsilon(x_0)\{\mathbf{l}^*, \mathbf{l}\}(x) &= \int dQ\, \epsilon(Q)Q\vartheta(Q^2 - m^2)\, e^{-|\vec{x}Q|} \\
&= \frac{1+|\vec{x}m|}{\vec{x}^2}\, e^{-|\vec{x}m|}
\end{aligned}
$$

$e^{-|\vec{x}Q|}$ in the integrand is a matrix element for the representation of the boost group $\mathbf{SO}^+(1, 1) \cong \mathbf{D}(1)$. The well-known Yukawa singularity structure

$1/|\vec{x}|$, $1/\vec{x}^2$ arising after integration with the spectral functions $\vartheta(Q^2 - m^2)$, $Q^2\vartheta(Q^2 - m^2)$ for the tangent appropriate particle fields cannot occur in $\mathbf{SQ}^+(1, 1)$ representations. A quantum representation of the spacetime model $\mathbf{D}(2)$ cannot be achieved alone by particle fields, genuine spacetime nonparticle field contributions have to occur.

3.3. Residual Representations for U(1) and D(1)

The Lie algebra $\log \mathbf{U}(n) \cong i\mathbb{R}(n)$ and the spacetime translations $\mathbb{R}(n)$ are unitarily diagonalizable with n Cartan coordinates in the polar decomposition:

$$\mathbb{R}(n) \cong \mathbb{R}^n \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1}: \qquad \begin{cases} i\alpha = u(\alpha) \circ i \operatorname{diag} \alpha \circ u(\alpha)^* \\ x = u(x) \circ \operatorname{diag} x \circ u(x)^* \end{cases}$$

For example,

$$n = 2: \quad i \operatorname{diag} \alpha = \begin{pmatrix} i(\alpha_0 + |\vec{\alpha}|) & 0 \\ 0 & i(\alpha_0 - |\vec{\alpha}|) \end{pmatrix},$$
$$\operatorname{diag} x = \begin{pmatrix} x_0 + |\vec{x}| & 0 \\ 0 & x_0 - |\vec{x}| \end{pmatrix}$$

leading to the Lie group and spacetime manifold as manifold products of a maximal Albelian Cartan subgroup and a compact submanifold:

$$\mathbf{U}(n) \cong \mathbf{U}(1)^n \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1}: \qquad e^{i\alpha} = u(\alpha) \circ e^{i \operatorname{diag} \alpha} \circ u(\alpha)^*$$
$$\mathbf{D}(n) \cong \mathbf{D}(1)^n \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1}: \qquad e^{x} = u(x) \circ e^{\operatorname{diag} x} \circ u(x)^*$$

Corresponding manifold products hold for the boost structure and the simple Lie symmetry

$$\mathbf{SU}(n) \cong \mathbf{U}(1)^{n-1} \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1}: \qquad \operatorname{tr} \alpha = 0$$
$$\mathbf{SD}(n) \cong \mathbf{D}(1)^{n-1} \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1}: \qquad \operatorname{tr} x = 0$$

The Cartan subsymmetry for the compact groups $\mathbf{SU}(n)$ and $\mathbf{U}(n)$ given by $\mathbf{U}(1)$-powers (tori) has its analogue in the $\mathbf{D}(1)$-powers (planes) as noncompact Cartan subsymmetry for the boost and causal symmetric spaces $\mathbf{SD}(n)$ and $\mathbf{D}(n)$.

The unitary irreducible representations of the Abelian group $\mathbf{GL}(\mathbb{C}_\mathbb{R}) = \mathbf{D}(1) \times \mathbf{U}(1)$, necessarily one-dimensional, have to be in $\mathbf{U}(1)$ since there is only one unitarity type in $\mathbf{GL}(\mathbb{C}_\mathbb{R})$. They must have an imaginary weight for the noncompact group $\mathbf{D}(1) \cong \mathbb{R}$ and an integer winding number for the periodic phase group $\mathbf{U}(1) \cong \mathbb{R}/\mathbb{Z}$,

$$\mathbf{D}(1) \times \mathbf{U}(1) \to \mathbf{U}(1) \subset \mathbf{GL}(\mathbb{C}_{\mathbb{R}})$$

$$e^{t+i\alpha} \mapsto e^{t\delta + i\alpha z} \Rightarrow \begin{cases} \overline{\delta} = -\delta = im \in i\mathbb{R} \\ z \in \mathbb{Z} \end{cases}$$

which leads to the representation weights, identical with the invariants,

$$\mathbf{weights}\ \mathbf{GL}(\mathbb{C}_{\mathbb{R}}) = \mathbf{weights}\ \mathbf{D}(1) \times \mathbf{weights}\ \mathbf{U}(1) = \{(im, z)\} = i\mathbb{R} \times \mathbb{Z}$$

An irreducible representation of the complex group $\mathbf{GL}(\mathbb{C})$ arises as a residue of its eigenvalue as singularity by using the complex Lie algebra forms $Q \in \mathbb{C}$,

$$\mathbf{GL}(\mathbb{C}) \ni e^{Z} \mapsto e^{Z\zeta} = \frac{1}{2i\pi} \oint dQ \frac{1}{Q - \zeta} e^{ZQ}, \qquad \zeta \in \mathbb{C}$$

which gives for the unitary irreducible $\mathbf{U}(1)$ and $\mathbf{D}(1)$-representations

$$\mathbf{U}(1) \ni e^{i\alpha} \mapsto e^{i\alpha z} = \frac{1}{2i\pi} \oint dw \frac{1}{w - z} e^{i\alpha w}, \qquad z \in \mathbb{Z}$$

$$\mathbf{D}(1) \ni e^{t} \mapsto e^{tim} = \frac{1}{2i\pi} \oint dq \frac{1}{q - m} e^{tiq}, \qquad im \in i\mathbb{R}$$

The integration for the noncompact and compact groups are related to each other for the Lie algebras and their forms,

$$\text{for } \mathbf{GL}(\mathbb{C}) \qquad (Z, Q), \quad Z = t + i\alpha, \quad Q = q + iw$$

$$\text{for } \mathbf{D}(1) \qquad (t, q) \leftrightarrow (i\alpha, iw) \text{ for } \mathbf{U}(1)$$

The nontrivial irreducible representations of $\mathbf{U}(1)$ and $\mathbf{D}(1)$ are not self-dual.

Measured representations use measures of the weights. The integer weights for the compact group $\mathbf{U}(1)$ have as discrete complex measures series of complex numbers leading to Fourier series:

$$\mathbf{meas}\ \mathbb{Z} \ni \{\mu_z\}_{z \in \mathbb{Z}} \mapsto \mathbf{rep}\ \mathbf{U}(1), \quad \mu_z \in \mathbb{C}$$

$$\mathbf{U}(1) \ni e^{i\alpha} \mapsto \sum_{z \in \mathbb{Z}} \mu_s e^{i\alpha z}$$

The continuous weights for $\mathbf{D}(1)$ have Lebesque measure dm-based complex measures giving rise to Fourier integrals,

$$\mathbf{meas}\ \mathbb{R} \ni \mu \mapsto \mathbf{rep}\ \mathbf{D}(1)$$

$$\mathbf{D}(1) \ni e^{t} \mapsto \int dm\, \mu(m) e^{tim}$$

The unitary irreducible representations of the simple group $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$ are characterized by self-dual representations of a Cartan subgroup,

$$\mathbf{GL}(\mathbb{C}_{\mathbb{R}})\sigma^3 = \mathbf{D}(1)\sigma^3 \times \mathbf{U}(1)\sigma^3 \cong \mathbf{SO}^+(1, 1) \times \mathbf{SO}(2)$$

which can go in the two types of two-dimensional unitary groups, the definite unitary **SU**(2) or the indefinite unitary **SU**(1, 1):

$$\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}}) \supset \mathbf{D}(1)\sigma^3 \times \mathbf{U}(1)\sigma^3 \to \left\{ \begin{matrix} \mathbf{U}(1)\sigma^3 \subset \mathbf{SU}(2) \\ \mathbf{D}(1)\sigma^3 \subset \mathbf{SU}(1, 1) \end{matrix} \right\} \subset \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$$
$$e^{(x_3 + i\alpha_3)\sigma^3} \to e^{(x_3\delta_3 + i\alpha_3 z_3)\sigma^3}$$

This defines the weights (δ_3, z_3) of the principal and supplementary series for **SU**(2) and **SU**(1, 1), respectively:

$$\mathbf{weights}^{(2,0)}\, \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}}) = \{(im_3, z_3)\} = i\mathbb{R} \times \mathbb{Z} = \mathbf{weights}\ \mathbf{GL}(\mathbb{C}_{\mathbb{R}})$$
$$\mathbf{weights}^{(1,1)}\, \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}}) = \{(m_3, 0)\} = \mathbb{R}$$

The principal series $\mathbf{GL}(\mathbb{C}_{\mathbb{R}})\sigma^3$-weights coincide with the $\mathbf{GL}(\mathbb{C}_{\mathbb{R}})$-weights. One $\mathbf{GL}(\mathbb{C}_{\mathbb{R}})\sigma^3$-representation is characterized by a dual pair $\{\pm im_3\}$ for $\mathbf{D}(1)\sigma^3$ and $\{\pm z_3\}$ for $\mathbf{U}(1)\sigma^3$. The new real $\mathbf{D}(1)\sigma^3$-weights $m_3 \in \mathbb{R}$ in contrast to the imaginary **D**(1)-weights $im \in i\mathbb{R}$ above are possible for dimensions $n \geq 2$ with the possibility of indefinite unitary groups. One $\mathbf{SO}^+$ (1, 1)-representation in **SU** (1, 1) is characterized by a dual pair $\{+m_3\}$. For dimensions $n \geq 3$ no additional types of invariants arise for the representations of the Cartan subgroups **U**(1) and **D**(1). Altogether the unitary **U**(1) and **D**(1)-representations are characterizable by the invariants

$$\mathbf{irrep}\ \mathbf{U}(1) \oplus \mathbf{irrep}\ \mathbf{SO}(2) \cong \{z\} \oplus \{2J\} = \mathbb{Z} \oplus \mathbb{N}$$
$$\mathbf{irrep}\ \mathbf{D}(1) \oplus \mathbf{irrep}\ \mathbf{SO}^+(1, 1) \cong \{im\} \oplus \{-m^2\} = i\mathbb{R} \oplus \mathbb{R}^-$$

Generalized functions have to be given taking care of the quadratic invariants as complex plane singularities for self-dual residual representations.

Pairs of dual irreducible **U**(1)-representations $\{e^{\pm i\alpha m} \mid m \in \mathbb{Z}\}$ can be formulated by measures with the integration prescription $m^2 \pm io = (|m| \pm io)^2$ for the invariant,

$$e^{\pm i|\alpha m|} = \pm \frac{1}{i\pi} \int dw \frac{|m|}{w^2 \mp io - m^2} e^{i\alpha w}, \qquad m \in \mathbb{R}$$

If the Cartan subgroup **U**(1) comes in the special group **SU**(n), $n \geq 2$, the residual representation employs the forms of the $\mathbb{R}^{n^2-1}$-dimensional tangent Lie algebra with the singularity of the generalized functions determined by the values of the invariant multilinear forms, starting for $n = 2$ with the bilinear Killing form $\vec{q}^2$ and a dipole

$$\text{for} \quad \mathbf{U}(1)\sigma^3 \cong \mathbf{SO}(2): \qquad e^{\pm i|\vec{\alpha}m|} = \pm\frac{1}{i\pi^2}\int d^3w \frac{|m|}{(\vec{w}^2 - m^2 \mp io)^2}\, e^{i\vec{\alpha}\vec{w}}, \qquad m \in \mathbb{R}$$

$$\textbf{irrep } \mathbf{SO}(2) \cong \{|m| = 2J\} = \mathbb{N}$$

Pairs of dual irreducible $\mathbf{D}(1)$-representations $\{e^{\pm xm}|m \in \mathbb{R}\}$ are obtained by $(i\alpha, iw) \leftrightarrow (x, q)$, leading to the following Lie algebra form measure:

$$e^{-|xm|} = \frac{1}{\pi}\int dq \frac{|m|}{q^2 + m^2}\, e^{-xiq}, \qquad m \in \mathbb{R}$$

For a boost manifold $\mathbf{SD}(n)$, $n \geq 2$, the $\mathbf{D}(1)\sigma^3$-representations use the $\mathbb{R}^{n^2-1}$-dimensional tangent space forms (momenta), again with a dipole for $n = 2$:

$$\text{for} \quad \mathbf{D}(1)\sigma^3 \cong \mathbf{SO}^+(1, 1): \qquad e^{-|\vec{x}m|} = \frac{1}{\pi^2}\int d^3q \frac{|m|}{(\vec{q}^2 + m^2)^2}\, e^{-\vec{x}i\vec{q}}, \qquad m \in \mathbb{R}$$

$$\textbf{irrep } \mathbf{SO}^+(1, 1) \cong \{-m^2\} = \mathbb{R}^-$$

3.4. Residual Representations for Spin SU(2)

The matrix elements of the irreducible $\mathbf{SU}(2)$-representations $[2J]$ by unitary $\mathbb{C}^{2J+1}$-automorphisms can be given via measures of the Lie algebra forms supported by integers.

With the generalized function singularities as angular momenta values,

$$\begin{aligned}\mathbf{SU}(2) &\cong \mathbf{SU}(2)/\mathbf{U}(1) \times \mathbf{U}(1)\sigma^3\\ &\cong \mathbf{SO}(3)/\mathbf{SO}(2) \times \mathbf{SO}(2)\end{aligned}$$

$$[\pm 1](\vec{\alpha}) = \frac{1}{\pi^2}\int d^3w \frac{\vec{w}}{(\vec{w}^2 - 1 \mp io)^2}\, e^{i\vec{\alpha}\vec{w}} = i\frac{\vec{\alpha}}{|\vec{\alpha}|}\, e^{\pm i|\vec{\alpha}|}$$

there arise the matrix elements of the *fundamental Pauli representation*

$$e^{i\vec{\alpha}\vec{\sigma}} = \mathbf{1}_2 \cos|\vec{\alpha}| + i\frac{\vec{\sigma}\vec{\alpha}}{|\vec{\alpha}| \sin|\vec{\alpha}|}$$

Using the irreducible $\mathbf{SO}(3)$-polynomials $[\vec{w}]^{2J}$, homogeneous of degree $2J$ in the angular momenta

$$[\vec{w}]^0 = 1, \qquad [\vec{w}]^1 = \{\vec{w}_a|a = 1, 2, 3\}, \qquad [\vec{w}]^2 = \left\{w_a w_b - \frac{\delta_{ab}}{3}\vec{w}^2\right\}, \ldots$$

the residual formulation for the matrix elements of the nontrivial *irreducible* $\mathbf{SU}(2)$-*representation* reads

$$\mathbf{SU}(2) \ni e^{i\vec{\alpha}\vec{\sigma}} \mapsto [\pm 2J](\vec{\alpha})$$
$$= \frac{1}{\pi^2}\int d^3w \frac{[\vec{w}]^{2J}}{(\vec{w}^2 - 4J^2 \mp io)^{2+J-c(J)}} e^{i\vec{\alpha}\vec{w}}, \qquad 2J = 1, 2, \ldots$$

The **SU**(2)-centrality (two-ality) $2c(J)$ is trivial for integer J and 1 for half-integer J

$$2c(J) = \begin{cases} 0, & 2J = 0, 2, 4, \ldots \\ 1, & 2J = 1, 3, \ldots \end{cases}$$

All representation elements of **SU**(2) can be obtained by derivations with respect to the invariant m^2 and the Lie parameter $\vec{\alpha}$ from the *Yukawa potential for* **SU**(2), defined in analogy to the usual Yukawa potential (next section), which is no **SU**(2)-representation because of the Lie parameter $\vec{\alpha} = 0$ singularity,

$$\frac{1}{\pi^2}\int d^3w \frac{1}{\vec{w}^2 - m^2 \mp io} e^{ig\vec{a}\vec{w}} = 2\frac{e^{\pm i|\vec{\alpha}m|}}{|\vec{\alpha}|}, \qquad m \in \mathbb{R}, \quad \vec{\alpha} \neq 0$$
$$\frac{\partial}{\partial m^2} = \frac{1}{2|m|}\frac{\partial}{\partial |m|}, \qquad \frac{\partial}{\partial \vec{\alpha}} = \frac{\vec{\alpha}}{\vec{\alpha}}\frac{\partial}{\partial |\vec{\alpha}|}$$

The m^2 derivative leads to

$$\frac{1}{\pi^2}\int d^3w \frac{1}{(\vec{w}^2 - m^2 \mp io)^2} e^{i\vec{\alpha}\vec{w}} = \pm i\frac{e^{\pm i|\vec{\alpha}m|}}{|m|}, \qquad m \in \mathbb{R}, \quad m \neq 0$$

which gives the trivial representation $[0](\vec{\alpha}) = 1$ for an appropriate limit $m \to 0$.

The representation matrix elements come in a product of a $\mathbf{U}(1)\sigma^3$-representation factor with the invariant $2J$ (rotation frequency) multiplying the modulus of the Lie parameter $|\vec{\alpha}|$ and a polynomial in the Lie parameter direction $\vec{\alpha}/|\vec{\alpha}|$ (rotation axis), homogeneous of degree $2J$, representing the two-dimensional symmetric space (2-sphere) **SU**(2)/**U**(1) $\cong$ **SO**(3)/**SO**(2),

$$[\pm 2J](\vec{\alpha}) \sim |\vec{\alpha}|^{1-2c(J)}\left[\frac{i\vec{\alpha}}{|\vec{\alpha}|}\right]^{2J} e^{\pm i2J|\vec{\alpha}|}$$

e.g., the adjoint representation

$$\mathbf{SO}(3){:}\quad [\pm 2](\vec{\alpha}) = \frac{1}{\pi^2}\int d^3w \frac{w_a w_b - \frac{1}{3}\delta_{ab}\vec{w}^2}{(\vec{w}^2 - 4 \mp io)^3} e^{i\vec{\alpha}\vec{w}}$$
$$= -\frac{|\vec{\alpha}|}{4}\left(\frac{\alpha_a\alpha_b}{\vec{\alpha}^2} - \frac{\delta_{ab}}{3}\right)e^{\pm 2i|\vec{\alpha}|}$$

to be compared with the elements in the (3×3) matrix

$$\delta_{ab} \cos 2|\vec{\alpha}| + \frac{\alpha_a \alpha_b}{\vec{\alpha}^2} (1 - \cos 2|\vec{\alpha}|) + \epsilon_{abc} \frac{\alpha_c}{|\vec{\alpha}|} \sin 2|\vec{\alpha}|$$

A measured **SU**(2)-representation is a Fourier series as for **U**(1) where each term comes with a unique **SU**(2)/**U**(1)-polynomial,

$$\mathbf{meas}\ \mathbb{Z} \ni \{\mu_z\}_{z \in \mathbb{Z}} \mapsto \mathbf{rep}\ \mathbf{SU}(2), \qquad \mu_z \in \mathbb{C}$$

$$\mathbf{SU}(2) \ni e^{i\vec{\alpha}\vec{\sigma}} \mapsto \sum_{2J=0,1,\ldots} (\mu_{2J}[2J] + \mu_{-2J}[-2J])$$

3.5. Residual Representations for Boost SD(2)

The unitary representations of the globally symmetric space $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{SU}(2)$, called a boost manifold **SD**(2), will be defined via the polar decomposition in a noncompact Cartan group **D**(1), in contrast to the compact **U**(1) for **SU**(2), and a compact submanifold **SU**(2)/**U**(1), identical for **SU**(2) and **SD**(2):

$$\mathbf{SD}(2) = \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{SU}(2) \cong \mathbf{SU}(2)/\mathbf{U}(1) \times \mathbf{D}(1)\sigma^3$$

$$\cong \mathbf{SO}^+(1, 3)/\mathbf{SO}(3) \cong \mathbf{SO}(3)/\mathbf{SO}(2) \times \mathbf{SO}^+(1, 1)$$

by using the tangent space relations

$$\text{for} \quad \mathbf{SD}(2) \quad (\vec{x}, \vec{q}) \leftrightarrow (i\vec{\alpha}, i\vec{w}) \quad \text{for} \quad \mathbf{SU}(2)$$

With the momentum measure singularities for the tangent space forms at the representation invariant $-m^2$ one obtains the *fundamental* **SD**(2)-*representations*

$$[m^2; 1](\vec{x}) = \frac{1}{\pi^2} \int d^3q \frac{i\vec{q}}{(\vec{q}^2 + m^2)^2} e^{-\vec{x}i\vec{q}} = \frac{\vec{x}}{|\vec{x}|} e^{-|\vec{x}m|}, \qquad m \in \mathbb{R}$$

to be compared with

$$e^{\vec{x}|m|\vec{\sigma}} = \mathbf{1}_2 \cosh |\vec{x}m| + \frac{\vec{\sigma}\vec{x}}{|\vec{x}|} \sinh |\vec{x}m|$$

and, in general, with the **SO**(3)-irreducible momentum polynomials $[\vec{q}]^{2J}$, the *irreducible* **SD**(2)-*representations*

$$\mathbf{SD}(2) \ni e^{\vec{x}\vec{\sigma}} \mapsto [m^2; 2J](\vec{x})$$

$$= \frac{1}{\pi^2} \int d^3q \frac{[i\vec{q}]^{2j}}{(\vec{q}^2 + m^2)^{2+j-c(j)}} e^{-\vec{x}i\vec{q}}, \qquad m \in \mathbb{R}, \quad 2J = 0, 1, 2, \ldots$$

In contrast to the group **SU**(2), where the representations of the compact factors $\mathbf{U}(1)\sigma^3$ and **SU**(2)/**U**(1) have to be related to each other by the invariant

$2J$, in the symmetric space $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{SU}(2)$ the invariant m^2 of the noncompact Cartan group $\mathbf{D}(1)\sigma^3$-representation is not related to the degree $2J$ of the homogenous polynomial for the representation of the compact sphere $\mathbf{SO}(3)/\mathbf{SO}(2)$.

All $\mathbf{SD}(2)$-representations can be obtained by derivations $\partial/\partial m^2$ and $\partial/\partial\vec{x}$ from the *Yukawa potential*, which, by itself, is no $\mathbf{SD}(2)$-representation because of the $\vec{x} = 0$ singularity:

$$\frac{1}{\pi^2}\int d^3q \, \frac{1}{\vec{q}^2 + m^2} \, e^{-\vec{x}i\vec{q}} = 2\frac{e^{-|\vec{x}m|}}{|\vec{x}|}, \qquad m \in \mathbb{R}, \quad \vec{x} \neq 0$$

The *scalar representations*

$$[m^2; 0](\vec{x}) = \frac{1}{\pi^2}\int d^3q \, \frac{1}{(\vec{q}^2 + m^2)^2} \, e^{-\vec{x}i\vec{q}} = \frac{e^{-|\vec{x}m|}}{|m|}, \qquad m \in \mathbb{R}, \quad m \neq 0$$

are trivial for the sphere $\mathbf{SO}(3)/\mathbf{SO}(2)$. The analogue to the adjoint spin representation reads

$$[m^2; 2](\vec{x}) = \frac{1}{\pi^2}\int d^3q \, \frac{-q_a q_b + \frac{1}{3}\delta_{ab}\vec{q}^2}{(\vec{q}^2 + m^2)^3} \, e^{-\vec{x}i\vec{q}} = \frac{|\vec{x}|}{4}\left(\frac{x_a x_b}{\vec{x}^2} - \frac{\delta_{ab}}{3}\right) e^{-|\vec{x}m|}$$

All irreducible representations can be written as products

$$[m^2; 2J](\vec{x}) \sim |\vec{x}|^{1-2c(J)} \left[\frac{\vec{x}}{|\vec{x}|}\right]^{2J} e^{-|\vec{x}m|}$$

A measured $\mathbf{SD}(2)$-representation is a sum over the spin numbers $2J$ with measures μ_{2J} for the continuous invariants:

$$\mathbf{meas}\ \mathbb{N} \times \mathbb{R}^+ \ni \{\mu_{2J}\}_{2J\in\mathbb{N}} \mapsto \mathbf{rep}\ \mathbf{SD}(2)$$

$$\mathbf{SD}(2) \ni e^{\vec{x}\vec{\sigma}} \mapsto \sum_{2J=0,1,\ldots} \int_0^\infty dm^2 \, \mu_{2J}(m^2)[m^2; 2J](\vec{x})$$

$$= \sum_{2J=0,1,\ldots} \int_0^\infty dm^2 \, \mu_{2J}(m^2) \, \frac{1}{\pi^2}\int d^3q \frac{[i\vec{q}]^{2J}}{(\vec{q}^2 + m^2)^{2+J-c(J)}} \, e^{-\vec{x}i\vec{q}}$$

The two integrations in measured representations go over the tangent space forms $\int d^3q$ and the invariants $\int_0^\infty dm^2$ with the dimensions three and one of the symmetric space and a Cartan subgroup, respectively. For the measured $\mathbf{SU}(2)$-representation in the previous section the one-dimensional integration is replaced by a discrete sum.

3.6. Two Continuous Invariants for Spacetime

Since Yukawa, the unification of a time development, characterized by a particle mass $|m_0|$, with a position space interaction, characterized by a range $1/|m_3|$, in one spacetime Klein–Gordon equation with one mass

$$\left.\begin{array}{r} \left(\dfrac{d^2}{dt^2} + m_0^2\right)\dfrac{e^{i|tm_0|}}{2i|m_0|} = \delta(t) \\ \left(-\dfrac{\partial^2}{\partial \vec{x}^2} + m_3^2\right)\dfrac{e^{-|\vec{x}m_3|}}{4\pi|\vec{x}|} = \sigma(\vec{x}) \end{array}\right\} \Rightarrow \begin{array}{l} (\partial^2 + m^2)G(x) = \delta(x) \\ \text{with } m_0^2 = m_3^2 = m^2 \end{array}$$

seems to be an obvious relativistic bonus.

Particle fields with a Dirac energy-momentum measure in their quantization

$$\mathbf{c}_j(x|m_0) = \int \frac{d^4q}{(2\pi)^3}\, \epsilon(q_0)q_j\delta(q^2 - m_0^2)e^{xiq}$$

give by position space integration a Dirac measure for the time weights $iq_0 \in i\mathbb{R}$ (real energies q_0), self-dually supported at $\pm im_0$, leading to $\mathbf{SO}(2)$-representation matrix elements of the Abelian time group $\mathbf{D}(1)$,

$$\int d^3x\, \mathbf{c}_j(x|m_0) = \delta_j^0 \int d^1q\, \epsilon(q)q\delta(q^2 - m_0^2)\, e^{x_0iq} = \delta_j^0 \cos x_0m_0$$

The appropriate measure for a representation of the boost subgroup $\mathbf{D}(1)\sigma^3 \cong \mathbf{SO}^+(1, 1)$ arises from a derived energy-momentum Dirac measure

$$\mathbf{c}_j^{\text{dip}}(x|m_3) = -\frac{d\mathbf{c}(x|m_3)}{dm_3^2} = \int \frac{d^4q}{(2\pi)^3}\, \epsilon(q_0)q_jd'(q^2 - m_3^2)e^{xiq}$$

Time integration leads to a Dirac measure for the $\mathbf{SO}^+(1, 1)$-invariant and an $\mathbf{SO}^+(1, 1)$-representation,

$$4\pi \frac{x_a}{|\vec{x}|}\delta_j^a \int dx_0\, \epsilon(x_0)\mathbf{c}_j^{\text{dip}}(x|m_3) = 2\int d^1q\, \epsilon(q)q\delta(q^2 - m_3^2)\, e^{-|\vec{x}q|} = e^{-|\vec{x}m_3|}$$

The appropriateness of the Dirac energy-momentum measure for time in contrast to the derived measure for position space

$$\left(-\frac{\partial^2}{\partial\vec{x}^2} + m_3^2\right)^2 \frac{e^{-|\vec{x}m_3|}}{3\pi|m_3|} = \delta(\vec{x})$$

reflects the different dimensions, one and three, respectively, as seen also in the energy-momentum Lebesque measure $d^4q = dq_0\, \vec{q}^{\,2}d|\vec{q}|\, d\varphi\, d\cos\theta$.

The association of the singularities at m_0^2 and m_3^2 to representation invariants for $\mathbf{D}(1)$ (time) and $\mathbf{SO}^+(1, 1)$, respectively, is blurred since a tangent space decomposition $\mathbf{x} = \mathbf{1}_2\mathbf{x}_0 + \vec{\sigma}\vec{x}$ into time and position space translations is not compatible with the action of the Lorentz group $\mathbf{SO}^+(1, 3)$. The Dirac measure also has a nontrivial projection for the boost $\mathbf{SO}^+(1, 1)$ and the derived Dirac measure a nontrivial projection for time $\mathbf{D}(1)$:

$$4\pi \frac{x_a}{|\vec{x}|}\delta_j^a \int dx_0 \, \epsilon(x_0)\mathbf{c}_j(\mathrm{x}|m_0) = 2\frac{1 + |\vec{x}m_0|}{\vec{x}^2} e^{-|\vec{x}m_0|}$$

$$\int d^3x \, \mathbf{c}_j^{\mathrm{dip}}(x|m_3) = \delta_j^0 \frac{x_0 \sin x_0 m_3}{2m_3}$$

The $\mathbf{D}(1)$-projection of the derived Dirac measure leads to matrix elements of reducible nondecomposable time representations.[13] The boost projection of the Dirac measure leads to a Yukawa force which is not related to a matrix element of an $\mathbf{SO}^+(1, 1)$-representation.

An ordered integration $d^4q \, \epsilon(q_0)$ with an energy-momentum Dirac measure coincides with an integration with an energy-momentum principal value, P, pole measure as shown by the identities

$$\int d^4q \, \epsilon(x_0 q_0)\delta^{(N)}(m^2 - q^2)e^{xiq}$$

$$= \frac{1}{i\pi}\int d^4q \, \frac{\Gamma(1 + N)}{(q_{\mathrm{P}}^2 - m^2)^{1+N}} e^{xiq}, \qquad N = 0, 1, \ldots$$

Related to two Cartan coordinates $x_0 \pm |\vec{x}|$ which reflect the real rank 2 of the noncompact homogeneous manifold $\mathbf{D}(2) = \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^2)/\mathbf{U}(2)$, i.e., two Abelian subgroups $\mathbf{D}(\mathbf{1}_2)$ (time) and $\mathbf{SO}^+(1, 1)$ as a subgroup of the boost manifold $\mathbf{SO}^+(1, 3)/\mathbf{SO}(3)$, two invariants are appropriate as support for the measures of the energy-momentum space with the action of the Lorentz group.

The unitary irreducible representations of the dilatation Lorentz group

$$\mathbf{GL}(\mathbb{C}_{\mathbb{R}}^2)/\mathbf{U}(\mathbf{1}_2) \cong \mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}^+(1, 3)$$

with Cartan subgroup $\mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}^+(1, 1) \times \mathbf{SO}(2)$ are characterized by two invariants (masses) from a continuous spectrum for the noncompact group $\mathbf{D}(\mathbf{1}_2) \times \mathbf{D}(1)\sigma^3$ (time and boost) and one possibly trivial integer invariant (winding number) for the compact polarization group $\mathbf{U}(1)\sigma^3$:

$$\mathbf{GL}(\mathbb{C}_{\mathbb{R}}^2)/\mathbf{U}(\mathbf{1}_2) \supset \mathbf{D}(\mathbf{1}_2) \times \mathbf{D}(1)\sigma^3 \times U(1)\sigma^3 \to \left\{ \begin{matrix} \mathbf{U}(2) \\ \mathbf{U}(1, 1) \end{matrix} \right\} \subset \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^2)$$
$$e^{x_0\mathbf{1}_2 + (x_3 + i\alpha_3)\sigma^3} \mapsto e^{x_0\delta_0\mathbf{1}_2 + (x_3\delta_3 + i\alpha_3 z_3)\sigma^3}$$

leading to the weights $(\delta_0, \delta_3, z_3)$ for principal and supplementary series:

$$\mathbf{weights}^{(2,0)}\mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{U}(\mathbf{1}_2) = \{(im_0, im_3, z_3)\} = i\mathbb{R} \times i\mathbb{R} \times \mathbb{Z}$$

$$\mathbf{weights}^{(1,1)}\mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{U}(\mathbf{1}_2) = \{(im_0, m_3)\} = i\mathbb{R} \times \mathbb{R}$$

The weights (im_0, m_3) of the supplementary series with trivial $\mathbf{SU}(2)$-representation are relevant for representations of spacetime $\mathbf{D}(2)$ as the unitary classes $\mathbf{D}(\mathbf{1}_2) \times \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{SU}(2)$. Here m_0 characterizes the positive unitary representations $\mathbf{D}(1) \ni e^{x_0} \mapsto e^{x_0 i m_0} \in \mathbf{U}(1)$ with a particle mass m_0 and a probability interpretation. m_3^2 characterizes the indefinite unitary representation $\mathbf{SO}^+(1, 1) \ni e^{-|\vec{x}|} \mapsto e^{-|\vec{x} m_3|} \in \mathbf{SU}(1, 1)$ with an interaction range $1/|m_3|$ and without particle asymptotics. There is no group-theoretic reason to identify both scales $m_0^2 = m_3^2$—in general, the representations of spacetime $\mathbf{D}(2)$ come with two different scales whose ratio m_3^2/m_0^2 is a physically important representation-characteristic constant.

The ratio of the characterizing invariants should be seen in analogy to the relative normalization of time and position space translations

$$\begin{pmatrix} l^2/c^2 & 0 \\ 0 & -l^2\mathbf{1}_3 \end{pmatrix}$$

as given with the maximal action velocity (speed of light) c^2.

3.7. Pole Measures of Energy-Momenta

To generalize the representations of the Abelian causal group $\mathbf{D}(1)$ as residues for energy singularities to representations of the homogeneous causal spacetime $\mathbf{D}(n)$ one starts from the matrix elements of nondecomposable representations of a Cartan subgroup $\mathbf{D}(1)^n$ with Cartan coordinates $\{\xi_r\}_{r=1}^n$, given as products of residues:

$$\mathbf{D}(1)^n \ni \begin{pmatrix} e^{\xi_1} & \dots & 0 \\ & \dots & \\ 0 & \dots & e^{\xi^n} \end{pmatrix} \mapsto \frac{(i\xi_1)^{N_1} \dots (i\xi_n)^{N_n}}{N_1! \dots N_n!} e^{\xi_1 i m_1 + \dots + \xi_n i m_n}$$

$$= \frac{1}{(2i\pi)^n} \oint d^n q \frac{e^{\xi_1 i q_1 + \dots + \xi_n i q_n}}{(q_1 - m_1)^{1+N_1} \dots (q_n - m_n)^{1+N_n}}$$

$$N_r = 0, \dots, N_r, \qquad r = 1, \dots, n$$

with real invariants $\{m_r\}_{r=1}^n$ (Cartan masses)and nildimensions $\{N_r\}_{r=1}^n$, trivial for the irreducible representations.

If the group $\mathbf{D}(1)^n$ comes as a Cartan subgroup in the spacetime manifold $\mathbf{D}(n)$

$$\mathbf{D}(1)^n \hookrightarrow \mathbf{D}(n) \cong \mathbf{D}(1)^n \times \mathbf{SU}(n)/\mathbf{U}(1)^{n-1}$$

one embeds in the Lebesque measure $d^{n^2}q$ of the energy-momenta

$$d^n q = d^1 q_1 \ldots d^1 q_n \quad \text{on} \quad \mathbb{R}^n \hookrightarrow d^{n^2} q \quad \text{on} \quad \mathbb{R}^{n^2}$$

invariant under $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$. The quotient with the nth power of the $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})$-invariant determinant (volume element)

$$\frac{d^{n^2} q}{(q^n)^n} \quad \text{with} \quad q^n = \det q = \begin{cases} q, & n = 1 \\ \det\begin{pmatrix} q_0 + q_3 & q_1 + iq_2 \\ q_1 - iq_2 & q_0 - q_3 \end{pmatrix}, & n = 2 \end{cases}$$

is a $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})$-invariant measure.

The $\mathbf{D}(1)^n$ eigenvalues are implemented as invariant singularities

$$\frac{d^n q}{(q_1 - m_1) \ldots (q_n - m_n)} \hookrightarrow \frac{d^{n^2} q}{(q^n - m_1^n) \ldots (q^n - m_n^n)}$$

leading to the *irreducible scalar pole measures* of the energy-momenta for $\mathbf{GL}(\mathbb{C}^n_{\mathbb{R}})$

$$\frac{d^{n^2} q}{(q^n - m_1^n) \ldots (q^n - m_n^n)} = \begin{cases} \dfrac{d^1 q}{q - m}, & n = 1 \\ \dfrac{d^4 q}{(q^2 - m_1^2)(q^2 - m_2^2)}, & n = 2 \end{cases}$$

Their invariance group is the homogeneous group $\mathbf{SL}(\mathbb{C}^n_{\mathbb{R}})/\mathbb{I}(n)$, i.e., $\mathbf{SO}^+(1, 3)$ for $n = 2$.

The compact manifold $\mathbf{SU}(n)/\mathbf{U}(1)^{n-1}$ with $2\binom{n}{2}$ coordinates can be nontrivially represented by energy-momentum polynomials.

3.8. Residual Representations of Spacetime

Matrix elements of Lie group representations can be formulated as residues for characterizing invariant singularities of their Lie algebra forms. This will be done also for representations of the real rank 2 symmetric spacetime $\mathbf{D}(2)$ using generalized functions of the noncompact $\mathbb{R}^4$-isomorphic energy-momenta $q \in \mathbb{R}(2)^T$ as linear forms of the $\mathbf{D}(2)$ tangent spacetime translations. Two energy-momentum invariants q^2 characterize the action of the causal and boost subgroup of $\mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})$.

Representations of spacetime

$$\begin{aligned} \mathbf{D}(2) &= \mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{U}(2) \\ &= \mathbf{D}(\mathbf{1}_2) \times \mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})/\mathbf{SU}(2) \cong \mathbf{D}(\mathbf{1}_2) \times \mathbf{D}(1)\sigma^3 \times \mathbf{SU}(2)/\mathbf{U}(1) \\ &\cong \mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}^+(1, 3)/\mathbf{SO}(3) \cong \mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}^+(1, 1) \times \mathbf{SO}(3)/\mathbf{SO}(2) \end{aligned}$$

will be built up by energy-momentum measures, compatible with the action of the Lorentz group $\mathbf{SO}^+(1, 3)$ on the tangent space. The $\mathbf{GL}(\mathbb{C}^2_{\mathbb{R}})$-invariant measures of the energy-momenta $d^4q/(q^2)^2$ use the $\mathbf{SL}(\mathbb{C}^2_{\mathbb{R}})$-invariant 2-form q^2 in the denominator. The two invariant masses characterizing the representations of a noncompact Cartan subgroup representation $\mathbf{GL}(\mathbb{C}^2_{\mathbb{R}}) \supset \mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}^+(1, 1) \to \mathbf{U}(1) \times \mathbf{SU}(1, 1)$ are implemented via singularities $d^4q/(q^2 - m_0^2)(q^2 - m_3^2)$ in the *irreducible spacetime representations*

$$\mathbf{D}(2) \ni e^x \mapsto [m_0^2, m_3^2; 2J](x) = \frac{1}{\pi^3} \int d^4q \frac{[q]^{2J}}{(q_P^2 - m_0^2)(q_P^2 - m_3^2)^{1+J+c(J)}} e^{xiq}$$

$$m_{0,3} \in \mathbb{R}, \qquad 2J = 0,1, \ldots$$

The spin-related factor

$$\frac{[q]^{2J}}{(q_P^2 - m_3^2)^{J+c(J)}}$$

with the centrality $2c(J) \in \{0, 1\}$ describes the Lorentz-compatible embedding of the sphere $\mathbf{SO}(3)/\mathbf{SO}(2)$ representations via the irreducible energy-momentum $\mathbf{SO}^+(1, 3)$-polynomials $[q]^{2J}$, homogeneous of degree $2J$:

$$[q]^0 = \{1\}, \qquad [q]^1 = \{q_j | j = 0, 1, 2, 3\}, \qquad [q]^2 = \left\{q_j q_k - \frac{\eta_{jk}}{4} q^2\right\}, \ldots$$

acted on by Lorentz group representations $[2J|2J]$. For nontrivial J the representations come with a multiple pole at m_3^2. As shown below, the spacetime representations depend on $\vartheta(x^2)x$, which reflects the manifold isomorphy of spacetime $\mathbf{D}(2)$ and the strictly positive cone $\{x \in \mathbb{R}^4 | \text{spec } x > 0\}$ in the tangent Minkowski spacetime translations.

The *measured spacetime representations*

$$\mathbf{meas}\ \mathbb{N} \times \mathbb{R}^{+2} \ni \{\mu_{2J}^0 \times \mu_{2J}^3\}_{2J \in \mathbb{N}} \mapsto \mathbf{rep}\ \mathbf{D}(2)$$

$$\mathbf{D}(2) \ni e^x \mapsto \sum_{2J=0,1,\ldots} \int_0^\infty dm_0^2\, dm_3^2\, \mu_{2J}^0(m_0^2)\mu_{2J}^3(m_3^2)[m_0^2, m_3^2; 2J](x)$$

involve a product measure for the continuous invariants $(m_0^2, m_3^2) \in \mathbb{R}^+ \times \mathbb{R}^+$.

The $\mathbf{D}(2)$-representations are different from the Lorentz-compatible position space distributions of time representations used for the quantization of tangent space particle fields (Källen–Lehmann representations[10]), e.g., for $2J = 1$

$$\text{particle fields:}\quad \int_0^\infty dm^2\ \mu(m^2)\,\frac{1}{\pi^3}\int d^4q\,\frac{q_j}{q_P^2 - m^2}\,e^{xiq},\qquad \mu(m^2)\geq 0$$

with positive-definite probability-related spectral measure $\mu(m^2)$ for the invariants of the time $\mathbf{D}(\mathbf{1}_2)$-representations in $\mathbf{U}(1)$.

The representations of rank 2 spacetime $\mathbf{D}(2)$ have to be seen as the generalization of measured representations for the rank 1 Abelian time group $\mathbf{D}(1)$

$$\mathbf{D}(1) \ni e^{t\mapsto}\int dm\ \mu(m)e^{tim} = \int dm\ \mu(m)\,\frac{\epsilon(t)}{i\pi}\int d^1q\,\frac{1}{q_P - m}\,e^{tiq}$$

The irreducible unitary time $\mathbf{D}(1)$-representations $e^{t\mapsto e^{tim}}$ use a Dirac energy measure with one supporting energy m. All matrix elements of the nondecomposable $\mathbf{D}(1)$-representations are given by derivatives with respect to the invariant,

$$\mathbf{D}(1) \ni e^{t\mapsto}(ti)^N e^{tim} = \frac{\epsilon(t)}{i\pi}\int d^1q\,\frac{\Gamma(1+N)}{(q_P - m)^{1+N}}\,e^{tiq} = \left(\frac{d}{dm}\right)^N e^{tim}$$

$$m\in\mathbb{R},\qquad N = 0, 1, \ldots$$

The spacetime analogue is given by the nondecomposable $\mathbf{D}(2)$-representation matrix elements with two supporting masses

$$\mathbf{D}(2) \ni e^{x\mapsto}\frac{1}{\pi^3}\int d^4q\,\frac{\Gamma(1+N_0)\Gamma(1+N_3)[q]^{2J}}{(q_P^2 - m_0^2)^{1+N_0}(q_P^2 - m_3^2)^{1+N_3+J+c(J)}}\,e^{xiq}$$

$$m_{0,3}\in\mathbb{R},\qquad 2J = 0, 1, \ldots,\qquad N_{0,3} = 0, 1, \ldots$$

which arise from the scalar irreducible ones $[m_0^2, m_3^2; 0]$ by derivations with respect to the invariants d/dm_0^2, d/dm_0^2, and—for the $\mathbf{SO}(3)/\mathbf{SO}(2)$ properties—by derivations with respect to the spacetime variable d/dx.

3.9. Cartan Group Projection of Spacetime Representations

The projection of the spacetime representations to representations of Cartan subgroups is given by *time* $\mathbf{D}(\mathbf{1}_2)$*-projection via position space integration* and *boost* $\mathbf{SO}^+(1, 1)$*-projection via time integration*:

$$\frac{\epsilon(x_0)}{8i\pi}\int d_3x:\qquad \textbf{rep D}(2)\to\textbf{rep D}\ (\mathbf{1}_2)$$

$$\frac{1}{2}\int dx_0:\qquad \textbf{rep D}(2)\to\textbf{rep SD}(2)$$

$$\left[\frac{\vec{x}}{|\vec{x}|}\right]^{2J}: \qquad \textbf{rep SD}(2) \to \textbf{rep SO}^+(1, 1)$$

The time projection for the irreducible representations

$$\frac{\epsilon(x_0)}{8i\pi}\int d^3x\,[m_0^2, m_3^2; 2J](x) = \frac{\epsilon(x_0)}{i\pi}\int d^1q \frac{[q_0]^{2J}}{(q_P^2 - m_0^2)(q_P^2 - m_3^2)^{1+J+c(J)}}\, e^{x_0 iq}$$

can be computed with the **SO**(2)-representation matrix elements

$$\frac{\epsilon(x_0)}{i\pi}\int d^1q \frac{\begin{pmatrix} q \\ m \end{pmatrix}}{q_P^2 - m^2}\, e^{x_0 iq} = \frac{1}{i\pi}\oint d^1q \frac{\begin{pmatrix} q \\ m \end{pmatrix}}{q^2 - m^2}\, e^{x_0 iq} = \begin{pmatrix} \cos x_0 m \\ i \sin x_0 m \end{pmatrix}$$

The energy-momentum polynomials are projected to energy polynomials

$$\text{with} \quad \begin{matrix} q_i \mapsto \delta_j^0 q_0 \\ g_{jk} \mapsto \delta_j^0 \delta_k^0 \end{matrix} \Rightarrow \begin{cases} [q_0]^0 = 1 \\ [q_0]^1 = q \\ [q_0]^2 = \frac{3}{4}(q_0)^2, \ldots \end{cases}$$

The projection to representations of the boost manifold

$$\frac{1}{2}\int dx_0\,[m_0^2, m_3^2; 2J](x) = \frac{1}{\pi^2}\int \frac{d^3q\,[q_0]^{2J}(-1)^{J+c(J)}}{(\vec{q}^2 + m_0^2)(\vec{q}^2 + m_3^2)^{1+J+c(J)}}\, e^{-\vec{x}i\vec{q}}$$

is computed with the Yukawa potential

$$\frac{1}{\pi^2}\int d^3q \frac{1}{\vec{q}^2 + m^2}\, e^{-\vec{x}i\vec{q}} = 2\frac{e^{-|\vec{x}m|}}{|\vec{x}|}, \qquad \vec{x} \neq 0$$

which by itself is no **SD**(2)-representation. The linear combinations occurring in the **SD**(2)-projection are measured **SD**(2)-representation with finite spectral moments for the measures, e.g.,

$$2\frac{e^{-|\vec{x}m_0|} - e^{-|\vec{x}m_3|}}{|\vec{x}|} = \int_{m_0^2}^{m_3^2} dm^2 \frac{e^{-|\vec{x}m|}}{|m|} = \int_0^\infty dm^2\,\mu_0(m^2)[m^2; 0](\vec{x})$$

$$\mu_0(m^2) = \vartheta(m^2 - m_0^2)\vartheta(m_3^2 - m_0^2), \qquad [m^2; 0](\vec{x}) = \frac{e^{-|\vec{x}m|}}{|m|}$$

$$\int_0^\infty dm^2\,\mu_0(m^2) = m_3^2 - m_0^2, \ldots$$

The irreducible energy-momentum polynomials are projected to momentum polynomials $[\vec{q}]^{2J}$, in general decomposable:

$$\text{with}\quad \begin{matrix} q_j \mapsto \delta_j^q q_a \\ q_{jk} \mapsto -\delta_j^q \delta_k^b \delta_{ab} \end{matrix} \Rightarrow \begin{cases} [q_a]^0 = 1 \\ [q_a]^1 = q_a \\ [q_a]^2 = q_a q_b - \dfrac{\delta_{ab}}{4} \vec{q}^2 = [\vec{q}]^2 + \dfrac{\delta_{ab}}{12} \vec{q}^2 [\vec{q}]^0, \ldots \end{cases}$$

3.10. Scalar Spacetime Representations

The *irreducible scalar spacetime representations* are

$$\mathbf{D}(2) \ni e^{x} \mapsto [m_0^2, m_3^2; 0](x) = \frac{1}{\pi^3} \int d^4 q \, \frac{1}{(q_P^2 - m_0^2)(q_P^2 - m_3^2)} e^{xiq}$$

The decomposition in energy-momenta measures with one singularity only

$$\frac{1}{(q^2 - m_0^2)(q^2 - m_3^2)} = \frac{1}{m_0^2 - m_3^2} \left[\frac{1}{q^2 - m_0^2} - \frac{1}{q^2 - m_3^2} \right]$$
$$\sim \frac{\delta(q^2 - m_0^2) - \delta(q^2 - m_3^2)}{m_0^2 - m_3^2}$$

gives the representation matrix elements for the time subgroup

$$\mathbf{D}(1) \ni e^{x_0} \mapsto \frac{\epsilon(x_0)}{8i\pi} \int d^3 x \, [m_0^2, m_3^2; 0](x) = \frac{i}{m_0^2 - m_3^2} \left[\frac{\sin x_0 m_0}{m_0} - \frac{\sin x_0 m_3}{m_3} \right]$$

The boost group $\mathbf{SO}^+(1, 1)$ is represented with $J = 0$,

$$\mathbf{SO}^+(1, 1) \ni e^{-|\vec{x}|} \mapsto \frac{1}{2} \int dx_0 \, [m_0^2, m_3^2; 0](x) = -2 \frac{e^{-|\vec{x} m_0|} - e^{-|\vec{x} m_3|}}{|\vec{x}|(m_0^2 - m_3^2)}$$

The explicit form of the irreducible scalar spacetime representations

$$[m_0^2, m_3^2; 0](x) = \vartheta(x^2) \frac{m_0^2 \varepsilon_1(m_0^2 x^2/4) - m_3^2 \varepsilon_1(m_3^2 x^2/4)}{m_0^2 - m_3^2}$$

with the special cases for equal and trivial masses

$$[m^2, 0; 0](x = \vartheta(x^2) \varepsilon_1(m^2 x^2/4)$$
$$[m^2, m^2; 0](x) = \vartheta(x^2) \varepsilon_0(m^2 x^2/4)$$
$$[0, 0; 0](x) = \vartheta(x^2)$$

contain the measured $\mathbf{D}(1)$-representations with Bessel functions J_k:

$$\varepsilon_k\left(\frac{\tau^2}{4}\right) = \frac{J_k(\tau)}{(\tau/2)^k} = \sum_{n=0}^{\infty} \frac{(-\tau^2/4)^n}{n!(n+k)!}$$

$$= \frac{1}{\sqrt{\pi}\Gamma(k+\frac{1}{2})} \int dE\, \sqrt{1-E^2}^{2k-1}\, \vartheta(1-E^2) e^{\tau i E}, \qquad k = 0, 1, \ldots$$

3.11. Fundamental Spacetime Representations

The irreducible *fundamental spacetime representations* belong to the generating real four-dimensional $\mathbf{SO}^+(1, 3)$-representation $[1|1]$,

$$\mathbf{D}(2) \ni e^{x} \mapsto [m_0^2, m_3^2; 1](x) = \frac{1}{\pi^3} \int d^4q\, \frac{q^j \sigma_j}{(q_\pi^2 - m_0^2)(q_P^2 - m_3^2)^2}\, e^{xiq}$$

They involve a simple pole (particle singularity) and a dipole (interaction singularity) reflecting the positive unitary and the indefinite unitary representation of a Cartan subgroup time $\mathbf{D}(1)$ and boost $\mathbf{SO}^+(1, 1)$, respectively.

The decomposition into energy-momenta measures with one singularity only

$$\frac{1}{(q^2 - m_0^2)(q^2 - m_3^2)^2} = \frac{1}{(m_0^2 - m_3^2)^2}\left[\frac{1}{q^2 - m_0^2} - \frac{1}{q^2 - m_3^2}\right]$$
$$- \frac{1}{(m_0^2 - m_3^2)}\, \frac{1}{(q^2 - m_3^2)^2}$$
$$\sim \frac{\delta(q^2 - m_0^2) - \delta(q^2 - m_3^2)}{(m_0^2 - m_3^2)^2} + \frac{\delta'(q^2 - m_3^2)}{m_0^2 - m_3^2}$$

gives the representation matrix elements for the time subgroup

$$\mathbf{D}(1) \ni e^{x_0} \mapsto - \frac{\epsilon(x_0)}{16\pi}\, \mathrm{tr} \int d^3x\, [m_0^2, m_3^2; 1](x)$$
$$= \frac{\cos x_0 m_0 - \cos x_0 m_3}{(m_0^2 - m_3^2)^2} + \frac{x_0 m_3 \sin x_0 m_3}{2m_3^2(m_0^2 - m_3^2)}$$

and those for the boost subgroup

$$\mathbf{SO}^+(1, 1) \ni e^{-|\vec{x}|} \mapsto \frac{1}{4i}\, \mathrm{tr}\, \frac{\vec{\sigma}\vec{x}}{|\vec{x}|} \int dx_0\, [m_0^2, m_3^2; 1](x)$$
$$= 2\frac{(1 + |\vec{x}m_0|)e^{-|\vec{x}m_0|} - (1 + |\vec{x}m_3|)e^{-|\vec{x}m_3|}}{\vec{x}^2(m_0^2 - m_3^2)^2} + \frac{e^{-|\vec{x}m_3|}}{m_0^2 - m_3^2}$$

The integrated form of the spacetime representation

$$[m_0^2, m_3^2; 1](x) = i\vartheta(x^2)\, x\, \frac{m_0^4 \varepsilon_2(x^2 m^2/4) - m_3^4 \varepsilon_2(x^2 m_3^2/4) - (m_0^2 - m_3^2) m_3^2 \varepsilon_1(x^2 m^2/4)}{2(m_0^2 - m_3^2)^2}$$

has the following special cases for equal and trivial masses

$$[m^2, 0; 1](x) = i\vartheta(x^2) x \frac{\varepsilon_2(m^2 x^2/4)}{2}$$

$$[0, m^2, 1](x) = i\vartheta(x^2) x \frac{\varepsilon_1(m^2 x^2/4) - \varepsilon_2(m^2 x^2/4)}{2}$$

$$[m^2, m^2; 1](x) = i\vartheta(x^2) x \frac{\varepsilon_0(m^2 x^2/4)}{4}$$

$$[0, 0; 1](x) = i\vartheta(x^2) x \tfrac{1}{4}$$

3.12. Spacetime Quantum Fields

Spacetime representations arise as field quantizations. In analogy to the time-dependent position $\mathbf{x}(t)$ quantized by a time $\mathbf{D}(1)$-representation matrix element, e.g., for the harmonic oscillator

$$[m^2](t) = \frac{1}{\pi} \int d^1 q \frac{1}{q_{\rm P}^2 - m^2} e^{tiq} = -\epsilon(t) \frac{\sin tm}{m} = i\epsilon(t)[x, x](t)$$

the $\mathbf{D}(2)$-spacetime residual representations are quantizations of spacetime fields, e.g., for the scalar and fundamental representations as commutator and anticommutator of a scalar and spinor field, respectively:

$$\begin{aligned}[m_0^2, m_3^2; 0](x) &= \frac{1}{\pi^3} \int d^4 q \frac{1}{(q_{\rm P}^2 - m_0^2)(q_{\rm P}^2 - m_3^2)} e^{xiq} \\ &= i\epsilon(x_0)[\Phi, \Phi](x) \\ [m_0^2, m_3^2; 1](x) &= \frac{1}{\pi^3} \int d^4 q \frac{q^j \sigma_j}{(q_{\rm P}^2 - m_0^2)(q_{\rm P}^2 - m_3^2)^2} e^{xiq} \\ &= i\epsilon(x_0)\{\Psi^*, \Psi\}\,(x)\end{aligned}$$

In contrast to time and because of the additional indefinite $\mathbf{SO}^+(1, 1)$ boost structure such spacetime fields cannot be interpreted in terms of positive metric particles only. Supplementing the residual spacetime representation which can be taken as a causally supported quantization in flat tangent spacetime by an Fock state value for the quantization opposite commutator, also spacelike supported,

$$\langle\{\Phi, \Phi\}(x) - \epsilon(x_0)[\Phi, \Phi](x)\rangle$$
$$= \frac{i}{\pi^3}\int d^4q \, \frac{1}{(q^2 + io - m_0^2)(q_{\mathrm{P}}^2 - m_3^2)} e^{xiq}, \qquad m_0^2 > m_3^2$$
$$\langle[\Psi^*, \Psi](x) - \epsilon(x_0)\{\Psi^*, \Psi\}(x)\rangle$$
$$= \frac{i}{\pi^3}\int d^4q \, \frac{q^j\sigma_j}{(q_{\mathrm{P}}^2 + io - m_0^2)(q_{\mathrm{P}}^2 - m_3^2)^2} e^{xiq}$$

only the m_0^2-singularity with a positive residue allows a particle interpretation and therefore an additional on-shell spacelike contribution as included with the integration prescription $+io$.

Starting from a frequency m for the creation operator u of a harmonic oscillator, the freqencies nm for the powers u_n with natural n arise as singularities by convolutions of the basic representations e^{tim} in the residual representation. Similarly, a fundamental spacetime representation $[m_0^2, m_3^2, 1]$, may give rise to product representations whose positive metric singularities have a particle interpretation. To this end the convolution, appropriate for the Abelian time group $\mathbf{D}(1) = \mathbf{GL}(\mathbb{C}_{\mathbb{R}})/\mathbf{U}(1)$, has to be generalized to a "convolution" for the non-Abelian spacetime symmetric space $\mathbf{D}(2) = \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^2)/\mathbf{U}(2)$.

As another genuine spacetime feature the class property of the spacetime elements with the fixgroup $\mathbf{U}(2)$ has to be taken into account,

$$\mathbf{GL}(\mathbb{C}_{\mathbb{R}}^2) \cong \mathbf{GL}(\mathbb{C}_{\mathbb{R}}^2)/\mathbf{U}(2) \times \mathbf{U}(2) = \mathbf{D}(2) \times \mathbf{U}(2)$$

This noncompact-compact factorization can be connected with the external-internal dichotomy.[18]

REFERENCES

1. H. Boerner, *Darstellungen von Gruppen*, Springer, Berlin (1995).
2. N. Bourbaki, *Théorie des Ensembles*, Hermann, Paris (1957), Chapter 4.
3. N. Bourbaki, *Groupes et Algèbres de Lie*, Hermann, Paris (1968–1975), Chapters 4–8.
4. V. Fock, *Z. Phys.* **98** (1935), 145.
5. D. R. Finkelstein, *Quantum Relativity*, Springer, Berlin (1996).
6. W. Fulton and J. Harris, *Representation Theory*, Springer, Berlin (1991).
7. I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions V (Integral Geometry and Representation Theory)*, Academic Press, New York (1966).
8. I. M. Gelfand and M. A. Neumark, *Unitäre Darstellungen der klassischen Gruppen*, Akademie Verlag, Berlin (1957).
9. S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York (1978).
10. G. Källen, *Helv. Phys. Acta* **25** (1952), 417; H. Lehmann, *Nuovo Cimento* **11** (1954), 342.
11. H. Weyl, *Raum-Zeit-Materie*, Wissenschaftliche Buchgesellschaft, Darmstadt (1923).
12. E. P. Wigner, *Ann. Math.* **40** (1939), 149.
13. H. Saller, *Nuovo Cimento* **104B** (1989), 291.

14. H. Saller, *Nuovo Cimento* **108B** (1993), 603; **109B** (1993), 255.
15. H. Saller, R. Breuninger, and M. Haft, *Nuovo Cimento* **108A** (1995), 1225,
16. H. Saller, *Int. J. Theor. Phys.* **36** (1997), 2783.
17. H. Saller, The central correlations of hypercharge, isospin, colour and chirality in the standard model, hep-th/9802112, to be published.
18. H. Saller, *Int. J. Theor. Phys.* **37** (1998), 2333.